Tate uniformization theorem:

Def: Let Y be a rigid analytic k-space, and let Γ be a group of automorphism of Y. We say that Γ acts discontinuously if there exists an admissible covering $(Y_i)_{i\in I}$ of Y such that

$$\{\gamma \in \Gamma \mid \gamma Y_i \cap Y_i \neq \emptyset\} \text{ is finite } \forall\, i \in I.$$

Given Y and Γ above, we can define a rigid analytic k-space denoted by Y/Γ and a map $p: Y \to Y/\Gamma$ as follows:

① set : Y/Γ is the set-theoretic quotient

② G-topology : $U \subseteq Y/\Gamma$ admissible if $p^{-1}(U)$ is admissible open in Y.

$\{U_i \to U\}_{i\in I}$ admissible if $\{p^{-1}(U_i) \to p^{-1}(U)\}_{i\in I}$ is so.

③ Structure sheaf: $\forall$ admissible open $U \subseteq Y/\Gamma$, $\mathcal{O}_{Y/\Gamma}(U) = \mathcal{O}_Y(p^{-1}(U))^{\Gamma}$.

Example: for $\mathbb{G}_m = \operatorname{Spec}(k[T^{\pm}])$ and $\mathbb{G}_m^{an}$, $q \in k$ s.t. $0 < |q| < 1$, then with the admissible cover $Y_n = \{x \in \mathbb{G}_m^{an} : |q|^{n+1} \leq |x| \leq |q|^n\}$, we see that the action $q^{\mathbb{Z}} \curvearrowright \mathbb{G}_m^{an}$ is discontinuous.

One can show that $E_q := \mathbb{G}_m^{an}/q^{\mathbb{Z}}$ is a connected, smooth, proper rigid analytic k-space of dim 1.

Theorem: (Weierstrass' form)

Let $S_k(q) = \sum_{n\geq 1} \frac{n^k \cdot q^n}{1-q^n}$, which converges for $|q|<1$.

Define $a_4(q) = -S_3(q)$, $a_6(q) = -\frac{5a_3(q)+7a_5(q)}{12}$, and

E_q' by $y^2 + xy = x^3 + a_4(q)x + a_6(q)$ in $\mathbb{P}^2$

Then we have

$$E_q := \mathbb{G}_m^{an}/q^{\mathbb{Z}} \xrightarrow{\simeq} E_q'$$

$$u \longmapsto \begin{cases} (X(q,u), Y(q,u)) & \text{if } u \notin q^{\mathbb{Z}} \\ 0 : \text{point at } \infty & \text{if } u \in q^{\mathbb{Z}} \end{cases}$$

where

$$X(q,u) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2S_1(q)$$

$$Y(q,u) = \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + 2S_1(q)$$

Theorem (Tate's uniformization of elliptic curves)

Let E be an elliptic curve over a non-archimedean field k.
After a suitable finite separable extension of k, then

① if $|j(E)|>1$, then $E \xrightarrow{\simeq} \mathbb{G}_m^{an}/q^{\mathbb{Z}}$ for a unique $q \in k^\times$, $0<|q|<1$.

The function $j(q) = \frac{(1+48a_4(q))^3}{\Delta(q)}$, for discriminant $\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}$

defines a biholomorphic map :

$$j : \{q \in k^\times : 0<|q|<1\} \xrightarrow{\simeq} \{j \in k^\times : |j|>1\}$$

② if $|j(E)|\leq 1$, then E has good reduction , i.e. reduce to an elliptic curve over the residue field and no uniformization at all

[c.f. Lütkebohmert, rigid geometry of curves and their Jacobians]
BGR, Chapter 9

Tate's rigid analytic spaces have the same points as k^n, but changes the topology to have a sheaf theory. But by carefully studying the following example, we'll construct a sheaf $\mathcal{F} \neq 0$ but its stalks are all zero, which suggests that we are missing points in $Sp(k\langle T \rangle)$ $\rightsquigarrow$ Berkovich spaces.

Example 1: (affinoid subdomain in a closed unit disk)

Consider the closed unit disk $B^1 = Sp(k\langle T\rangle)$, $k = \bar{k}$.
We will construct a sheaf $\mathcal{F} \neq 0$ such that its stalks are zero.
First, let's understand to open sets better.

Def: a subset X in B^1 is called standard if it's of the form

$$X = \{x \in B^1 : |x-a_1| \leq r, \ |x-a_2| \geq r_2, \ \dots, \ |x-a_n| \geq r_n\}$$

for some $a_i \in k$, $r_i \in |k|$, $0 \leq r_i \leq 1$.

Lemma: Any standard subset is an affinoid subdomain of B^1.
If X, Y are standard subsets s.t. $X \cap Y \neq \emptyset$, then $X \cup Y$ and $X \cap Y$ are standard subsets of B^1.

Theorem : (topology of a disk B^1)

① A subset $X \subseteq B^1$ is standard iff X is connected affinoid subdomain

② X is an affinoid subdomain of B' iff X is a finite disjoint union of of standard sets in B^1.

proof: we show that if X is an affinoid subdomain, then X is a finite disjoint union of standard sets in B^1:

① By the Gerritzen - Grauert Theorem, any affinoid subdomain is a finite union of rational subdomains. It suffices to show for

$$X = \{x \in B^1 : |f_1(x)| \le |g(x)|, \cdots, |f_i(x)| \le |g(x)|\}$$

② By the previous lemma, it suffices to consider the case $i = 1$, $|f(x)| \le |g(x)|$

③ Reduce to f, g polynomial:

By Weierstrass preparation theorem: $\exists !$ unit $e \in T_1$, $w \in k[x]$ s.t. $f = e \cdot w$. Same for g

Write $e = c \cdot (1 + s)$ where $c \in k$ and $s \in T_1$, $s(0) = 0$, and $|s| < 1$. Then $\forall x \in B^1$, $|e(x)| = |c|\,|1 + s(x)| = |c|$, which is a constant.

So it suffices to consider $X = \{x \in B^1 : |f(x)| \le |g(x)|\}$ where

$$f(x) = \prod_{i=1}^{n} (\xi - \alpha_i), \quad g(x) = \prod_{j=1}^{m} c\,(\xi - \beta_j)$$

for $c_i \neq \xi_j$, and f, g polynomials

④ Consider a circle of radius r:

$$\Gamma_r = \{x \in B^1 : |x| = r\}$$

Denote by $\|\cdot\|_r$ the supremum norm of the function on Γ_r.

Then $\|f\|_r = \prod_{i=1}^{n} \|\xi - \alpha_i\|_r = \prod_{i=1}^{n} \max(r, |\alpha_i|)$

So $\log \|f\|_r$ is a piecewise linear function and convex in $\log r$, and

$$R := \{ r \in |k| : 0 \le r \le 1 \text{ and } \|f\|_r \le \|g\|_r \}$$

is a finite union of closed intervals of form $\{ r \in |k| : r_1 \le r \le r_2 \}$.

So the set $S := \{ x \in B^1 : \|f\|_{|x|} \le \|g\|_{|x|} \}$ is a finite union of closed disk and closed annuli, thus standard open.

* if $r = |x| \notin \Delta = \{ |\alpha_1|, \cdots, |\alpha_n|, |\beta_1|, \cdots, |\beta_m| \} - \{0\}$,

$|f(x)| = \prod_i |x - \alpha_i| = \prod_i \max(|x|, |\alpha_i|) = \|f\|_r$

Similarly, $|g(x)| = \|g\|_r$

So for these x, $|f(x)| \le |g(x)|$ iff $\|f\|_r \le \|g\|_r$, so $x \in S$.

Let $S' = \{ x \in B^1 : |g(x)| = \|g\|_{|x|} \}$ which is a standard open.

Further, if $x \in S \cap S'$ then $|f(x)| \le \|f\|_{|x|} \le \|g\|_{|x|} = |g(x)|$ so $S \cap S' \subseteq X$

We have a cover of $X = \left(\bigsqcup_{x \notin \Delta} X \cap \Gamma_r \right) \cap \left(\bigcup_{x \in \Delta} X \cap \Gamma_r \right)$

$= \bigcup_{x \notin \Delta} (S \cap S' \cap \Gamma_r) \cap \left(\bigcup_{x \in \Delta} X \cap \Gamma_r \right)$

$= (S \cap S') \cap \left(\bigcup_{x \in \Delta} X \cap \Gamma_r \right)$

compute that each $X \cap \Gamma_r$ for $r \notin \Delta$ is covered by disjoint standard opens ▨

Define a sheaf on $k\langle T \rangle$, s.t. $\forall$ any affinoid subdomain $Y \subseteq X$, and a cover $Y = \bigsqcup U_i$, where U_i is a standard set. Set $d(Y) := \max_i d(U_i)$.

$\forall$ standard open set $U \subseteq X$, $F(U) = \begin{cases} \mathbb{Z} & \text{if } d(U) = 1 \\ 0 & \text{if } d(U) < 1 \end{cases}$

and an affinoid subdomain $Y = \bigsqcup U_i$, $F(Y) = \bigoplus_i F(U_i)$

$\rightsquigarrow$ this is a sheaf since $\forall$ affinoid subdomain $Y \subseteq X$, and finite cover $Y = \bigsqcup_{i=1}^{n} Y_i$, $d(Y) = \max(Y_i)$.

But F has zero stalks

Berkovich unit disk.

Theorem(Gelfand - Mazur)
Every multiplicative seminorm on $\mathbb{C}[T]$ extending usual absolute value/$\mathbb{C}$ is of the form $f \to |f(z)|$ for some $z \in \mathbb{C}$.

Assume k: complete, algebraically closed, non-archimedean valuation v.

$$A = k\langle T\rangle := \{ \sum_{i\geq 0} a_i T^i \text{ s.t. } \lim_{i\to\infty} |a_i| = 0 \}$$

① a **multiplicative semi-norm** on A is a function $|\ |_x : A \to \mathbb{R}_+$ s.t.

$|0|_x = 0$, $|1|_x = 1$, $|f\cdot g|_x = |f|_x \cdot |g|_x$, $|f+g|_x \leq |f|_x + |g|_x$ $\forall f.g \in A$

norm if $|f|_x = 0 \Rightarrow f = 0$

$\|f\|$ is Gauss norm ↓

② $|\cdot|_x$ is called **bounded** if $\exists$ a constant C_x s.t. $\forall f \in A$, $|f|_x \leq C_x \|f\|$.

Lemma: Let $|\cdot|_x$ be a bounded multiplicative seminorm on A. Then $\forall f.g \in A$,

(1) $|f|_x \leq \|f\|$ for all $f \in A$

(2) $|c|_x = |c|$

(3) $|f+g|_x \leq \max(|f|_x, |g|_x)$ and $|f+g|_x = \max(|f|_x, |g|_x)$ if $|f|_x \neq |g|_x$

pf (1) By boundedness, $|f|_x^n = |f^n|_x \leq C_x \|f^n\|_x = C_x \|f_x\|^n$, so $|f|_x \leq C_x^{\frac{1}{n}} \|f\|$

Take $n \to \infty$, $|f|_x \leq \|f\|$.

(2) if $c = 0$, true.

if $c \neq 0$, $[c]_x \leq \|c\| = |c|$, $[c^{-1}]_x \leq |c^{-1}|$, but $|c|\cdot|c^{-1}| = 1$, $\Rightarrow |c|_x = |c|$.

(3) $|f+g|_x^n = [(f+g)^n]_x \leq \max \left|\binom{n}{k}\right| \cdot |f|_x^k |g|_x^{n-k} \leq \max |f|_x^k |g|_x^{n-k}$

$\leq (\max |f|_x, |g|_x)^n$

So $|f+g|_x \leq \max(|f|_x, |g|_x)$

* a bounded multiplicative seminorm $[\ ]_x$ on A is like a nonarchimedean abs value on k.

Def: the Berkovich unit disk $D(0.1)$ is the functional analytic spectrum of $\mathcal{A} = k\langle T\rangle$, i.e. the set of all bounded multiplicative semi-norms on $k\langle T\rangle$.

It's equipped with Berkovich topology: the weakest topology s.t. $\forall f \in \mathcal{A}$ and $\alpha \in \mathbb{R}$, the sets

$$U(f,\alpha) := \{x \in D(0.1) : |f|_x < \alpha\}$$
$$V(f,\alpha) := \{x \in D(0.1) : |f|_x > \alpha\}$$

are open.

* $\forall$ element $f \in \mathcal{A}$ defines a continuous function: $f: D(0.1) \longrightarrow \mathbb{R}$, $x \longmapsto |f|_x$

Example of points on $D(0,1)$

① $\forall a \in D(0.1) = \{x \in k : |x| \leq 1\}$, it defines the valuation norm, i.e $\forall f \in \mathcal{A}$, $|f|_a = |f(a)|$

② $\forall$ closed disk $D(a,r) \subseteq D(0.1)$ in k, we have the supremum norm:

$$|f|_{D(a,r)} = \sup_{z \in D(a,r)} |f(z)|$$

It's not direct that $|f|_{D(a,r)}$ is multiplicative, which actually follows from the maximal modulus principle, i.e. if $\sum a_i (T-a)^i$ satisfies $\lim |a_i| r^i = 0$ the $|f|_{D(a,r)} = \sup |a_i| \cdot r^i$

③ For any decreasing sequence of discs $\{D(a_i, r_i)\}$, it defines the limit seminorm

$$|f|_x = \lim_{i \to \infty} |f|_{D(a_i, r_i)}$$

Theorem: (Berkovich classification for $D(0.1)$)

Every point $x \in D(0.1)$ can be realized as

$$|f|_x = \lim_{i\to\infty} |f|_{D(a_i,r_i)}$$

for some nested discs $D(a_1,r_1) \supseteq D(a_2,r_2) \supseteq \cdots$

If this sequence has a non-empty intersaction, then either

① the intersaction is a point, then $|f|_x = |f(a)|$, or

② $\bigcap_i D(a_i,r_i) = D(a.r)$, then $|f|_x = |f|_{D(a,r)}$.

* there are examples s.t. the sequence has empty intersaction, (e.g. $\mathbb{C}_p$). If all nested sequence of closed unit disk has non-empty intersaction, we say k is spherically complete. (e.g. $\mathbb{Q}_p$, $k((t))$)

* For Archimedean field, spherically complete $\Leftrightarrow$ complete. For non-Archimedean field, spherically complete $\Rightarrow$ complete. Any k can be embedding into a spherically complete field via "Hahn series."

So $D(0.1)$ has four types of points:

Type I: $|\cdot|_a$ corresponding to a point $a \in k$.

(One can show that if $\{D(a_i,r_i)\}$ satisfies $\lim r_i = 0$, the the completness of k implies $\bigcap D(a_i,r_i) = \{a\}$)

Type II: $\zeta_{a.r}$ = sup norm on $D(a,r) = \bigcap D(a_i,r_i)$, with $r>0$ belonging to the value group $|K|$

Type III: $\zeta_{a.r}$ = sup norm on $D(a,r) = \bigcap D(a_i,r_i)$, with $r>0$ doesn't belong to the value group

Type IV: $|f|_x = \lim_i |f|_{D(a_i,r_i)}$ with $\bigcap D(a_i,r_i) = 0$

Lemma: each $x \in D(0,1)$ is determined by its value on the linear polynomials.

For $a \in D(0,1) \subseteq k$, $r>0$, define the open and closed Berkovich discs:

$$D(a,r)^- = \{ x \in D(0,1) : |T-a|_x < r \}$$
$$D(a,r) = \{ x \in D(0,1) : |T-a|_x \leq r \}$$

Prop: A basis of open sets of $D(0,1)$ is given by the sets

$$D(a,r)^-, \quad D(a,r)^- \setminus \bigcup_{i=1}^{N} D(a_i,r_i), \quad D(0,1) \setminus \bigcup_{i=1}^{N} D(a_i,r_i).$$

for $a, a_i \in D(0,1)$, and $r, r_i \in |K^*|$.

Corollary: type I points $D(0,1) \subseteq k$ are dense in Berkovich unit disc.

Theorem. $D(0,1)$ is compact, Hausdorff, path-connected.

Tree structure on $D(0,1)$.

define a partial order on $D(0,1)$ s.t. $\forall\, x, y \in D(0,1)$,

$$x \le y \text{ iff } \forall f \in k\langle T\rangle,\ |f|_x \le |f|_y.$$

Recall that the Gauss norm $\zeta_{0,1}(\sum a_i T_i) = \max |a_i|$ satisfies

$$\forall\, x \in D(0,1),\quad |f|_x \le \|f\| \hookrightarrow \text{Gauss norm}$$

so $\zeta_{0,1}$ is the unique maximal point with respect to this order

Lemma:

① for type II, III points, $\zeta_{a,r} \le \zeta_{a',r'}$ iff $D(a,r) \subseteq D(a',r')$.

② More generally, for each $x, y \in D(0,1) \setminus \zeta_{0,1}$, represent them as strictly decreasing dics $\{D(a_i,r_i)\}$, $\{D'(a_i,r_i)\}$. Then $x \le y$ iff $\forall\, k \ge 0$, $\exists\, m, n \ge k$ s.t. $D(a_m, r_m) \subseteq D(a_n', r_n')$.

Corollary: type I and type IV points are minimal under this partial order.

pf: if x is type I and $y = \{D(a_i,r_i)\} \le x$, then $\exists\, k$, s.t. $\forall\, m \ge k$, $D(a_m, r_m) \subseteq \{x\}$, so $y = x$.

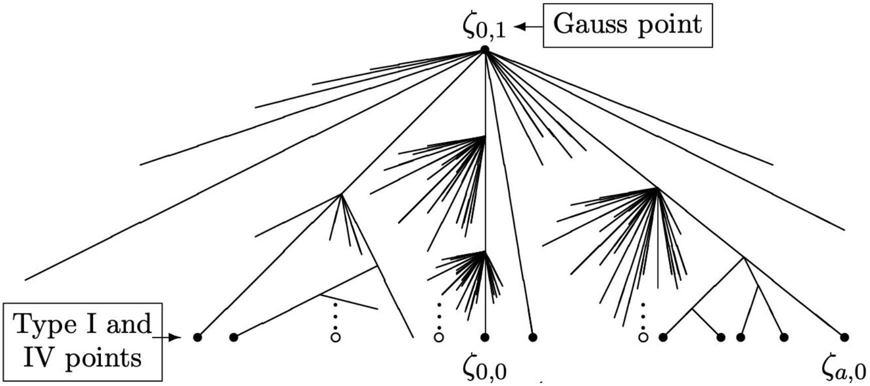
if x is type IV, $x = \{D(a_i,r_i)\}$, $y = \{D(a_i', r_i')\}$ and $y \le x$, then $\bigcap D(a_i,r_i) = 0$ for x, implies that $\bigcap D(a_i', r_i') \subseteq D(a_i, r_i) = \emptyset$ so y is also type IV.

Lemma: let x be a type I, II or III point, represented by $D(a,r)$. Then the map $[r,1] \longrightarrow D(0,1)$ sending t to $\zeta_{a,t} \in D(0,1)$ is a continuous embedding.

Similarly for type IV points $x = \{D(a_i,r_i)\}$ with $\lim r_i = r$, we have an embedding:

$$[r,1] \xrightarrow{\sim} [x, \zeta_{0,1}] := \{\text{all } \zeta_{a',r'} \in D(0,1) \text{ s.t. } D(a_i,r_i) \subseteq D(a,r) \text{ for some } i\}$$

Theorem: $D(0,1)$ is uniquely path-connected, Hausdorff and compact. It has a subbasis given by connected components of $D(0,1)\setminus\{x\}$, for $x \in D(0,1)$



Affine line $\mathbb{A}^{1,an}_k$

Consider $k\langle R^{-1}T\rangle := \{\sum_{k=0}^{\infty} c_k T^k \in k[\![T]\!] : \lim R^k |c_k| = 0\}$ formal power series having radius of convergence $\geq R$.

Define Berkovich disk of radius R:

$$D(0,R) := \mathcal{M}(k\langle R^{-1}T\rangle)$$

If $0<r<R$, we have a natural map $\pi: k\langle R^{-1}T\rangle \longrightarrow k\langle r^{-1}T\rangle$, whichs sends a seminorm $x \in D(0,r)$, to the seminorm: $(f \longrightarrow |\pi(f)|_x) \in D(0,R)$

This map is continuous and injective. We define $\mathbb{A}^{1,an}_k := \bigsqcup_{R>0} D(0,R)$

Prop: $\mathbb{A}^{1,an}_k$ is homeomorphic to all multiplicative seminorms on $k[T]$, with the weakest topology s.t. $\mathbb{A}^{1,an} \longrightarrow \mathbb{R}_+$ is continuous $\forall f \in k[T]$.
$x \longrightarrow |f|_x$

Thm : (Berkovich's classification for $\mathbb{A}^{1,an}_{k}$)

Every $x \in \mathbb{A}^1_{Berk}$ can be realized $|f|_x = \lim_{i\to\infty} |f|_{D(a_i,r_i)}$ for some nested disks in K: $D(a_1,r_1) \supseteq D(a_2,r_2) \supseteq \cdots$

If the intersaction is non-empty, then either

① $\bigcap_i D(a_i,r_i) = \{a\}$, for some $a \in k$, and $|f|_x = |f(a)|$ or

② $\bigcap_i D(a_i,r_i) = D(a,r)$ for $r>0$, and $|f|_x = \zeta_{a,r}$.

$\forall\ x \in \mathbb{A}^{1,an}_{k}$, define the local rring R_x of x in $k(T)$ by

$$R_x = \left\{ \frac{g}{h} \in k(T) : g, h \in k[T],\ |h|_x \neq 0 \right\}$$

we can extend $|\ |_x$ to a multiplicative seminorm on R_x by

$$\left|\frac{g}{h}\right|_x = \frac{|g|_x}{|h|_x}$$

Denote by $|R_x^*|_x$ the value group of $|\cdot|_x$, and write

$$\tilde{k}_x = \frac{\{f \in R_x : |f|_x \le 1\}}{\{f \in R_x : |f|_x < 1\}} \quad \text{the residue field at } x$$

Proposition : (intrinsic classification of points)

given $x \in \mathbb{A}^{1,an}_{k}$, then x is

type I iff $R_x \subsetneq k(T)$, $|R_x^*|_x = |K^\times|$, $\tilde{k}_x = \tilde{k}$

type II iff $R_x = k(T)$, $|R_x^*|_x = |K^\times|$, $\tilde{k}_x = \tilde{k}(t)$

type III iff $R_x = k(T)$, $|R_x^*|_x \supsetneq |K^\times|$, $\tilde{k}_x = \tilde{k}$

type IV iff $R_x = k(T)$, $|R_x^*|_x = |K^\times|$, $\tilde{k}_x = \tilde{k}$