

ξ : affinoid functions.

Now we define a (pre)-sheaf of affinoid functions on affinoid space $Sp(A)$.

Def: A (weak) Grothendieck topology \mathcal{J} (or a weak Γ -topology) consists of a category $\text{Cat } \mathcal{J}$, and a set $\text{Cov } \mathcal{J}$ of families of morphisms $(U_i \rightarrow U)_{i \in I}$ in $\text{Cat } \mathcal{J}$,

Objects in $\text{Cat } \mathcal{J}$ are called admissible open subsets in X , families in $\text{Cov } (\mathcal{J})$ are called admissible coverings.

such that

- ① if $U \rightarrow V$ is an isomorphism in $\text{Cat } \mathcal{J}$, then $(U \rightarrow V) \in \text{Cov } \mathcal{J}$
- ② if $(U_i \rightarrow U)_{i \in I} \in \text{Cov } \mathcal{J}$ and for each $i \in I$, $(V_{ij} \rightarrow U_i)_{j \in I_i} \in \text{Cov } \mathcal{J}$, then $(V_{ij} \rightarrow U)_{\substack{i \in I \\ j \in I_i}} \in \text{Cov } \mathcal{J}$
- ③ if $(U_i \rightarrow U)_{i \in I} \in \text{Cov } \mathcal{J}$ and $V \rightarrow U$ is a morphism in $\text{Cat } \mathcal{J}$, then $U_i \times_{\bigcup_j U_j} V$ exists for all $i \in I$ and $(U_i \times_{\bigcup_j U_j} V \rightarrow V)_{i \in I} \in \text{Cov } \mathcal{J}$

Example: for X a topological space, it has a Γ -topology by:

$\text{Cat } \mathcal{J} := \text{Open}(X)$, $\text{Cov } (\mathcal{J}) :=$ families of open coverings of open subsets

Example. for $X = \text{Sp}(A)$ an affinoid space, it has a G -topology by:

$\text{Cat } J :=$ affinoid subdomains and inclusions

$\text{Cov } J :=$ finite families $\{U_i \rightarrow U\}_{i \in I}$ of inclusions of affinoid subdomains in X such that $U = \bigcup_{i \in I} U_i$

We've shown admissible open are actually open in canonical topology.

Further, $\varphi: Z \rightarrow X$ affinoid, φ^{-1} takes admissible open and covering to admissible.

Def: Given a G -topology J on some space, a **presheaf** is a functor $(\text{Cat})^{\text{op}} \rightarrow \text{Set}$.

A presheaf is a **sheaf** if for every covering $(U_i \rightarrow U)_{i \in I}$ we have

$$F(U) \cong \text{Ker} \left(\prod_{i \in I} F(U_i) \rightarrow \prod_{i, j} F(U_i \times U_j) \right)$$

Example: For $X = \text{Sp}(A)$ an affinoid k -space, define a presheaf \mathcal{O}_X by:

$$\text{Cat } J \longrightarrow \text{Set}$$

affinoid subdomain $U \rightarrow \mathcal{O}_X(U)$ the associated affinoid algebra

Def: for any $x \in X$, the **stalk** of \mathcal{O}_X at x is

$$\mathcal{O}_{X,x} := \text{colim}_{\substack{\alpha \in U \subseteq X \\ \text{affinoid subdomain}}} \mathcal{O}_X(U)$$

Proposition:

- ① $\mathcal{O}_{X,x}$ is a local ring with the maximal ideal $\mathfrak{m}_x \cdot \mathcal{O}_{X,x}$
- ② The canonical map $A_{\mathfrak{m}_x} \rightarrow \mathcal{O}_{X,x}$ is injective, and $\hat{A} \xrightarrow{\cong} \hat{A}_{\mathfrak{m}_x} \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}$.
- ③ $\forall f \in \mathcal{O}_X(U)$, then $f=0$ iff $x \in U$, $[f] = 0$ in $\mathcal{O}_{X,x}$
- ④ $\mathcal{O}_{X,x}$ is noetherian.

Theorem (Tate acyclic theorems)

Let K be an affinoid k -space, and \mathcal{U} a finite covering of X by affinoid subdomains. Then \mathcal{U} is acyclic with respect to the presheaf \mathcal{O}_X of affinoid functions on X , i.e. the following sequence is exact

$$0 \rightarrow \mathcal{O}_X(\mathcal{U}) \rightarrow \prod_i \mathcal{O}_X(U_i) \rightarrow \prod_{i < j} \mathcal{O}_X(U_i \cap U_j) \rightarrow \dots$$

In particular, \mathcal{O}_X is a sheaf for the weak G -topology.

§ Global rigid analytic space.

Def: Let X be an affinoid k -space. The strong Grothendieck topology on X is given as follows:

(1) An subset $U \subseteq X$ is an **admissible open** if there is a covering $U = \bigcup_{i \in I} U_i$ by (maybe infinite) affinoid subdomains $U_i \subseteq X$ satisfying:
for all morphisms of affinoid k -spaces $\varphi: Z \rightarrow X$ with $\varphi(Z) \subseteq U$ the covering $\langle \varphi^{-1}(U_i) \rangle_{i \in I}$ of Z admits a refinement by finitely many affinoid domains.

(2) A covering $V = \bigcup_{j \in J} V_j$ of some admissible open $V \subseteq X$ by admissible open subsets V_j is admissible if:

\forall morphism $\varphi: Z \rightarrow X$ of k -affinoid spaces with $\varphi(Z) \subseteq V$, the covering $\langle \varphi^{-1}(V_j) \rangle_{j \in J}$ of Z admits a refinement that is a finite covering of Z by affinoid subdomains.

Example: (strong G -topology)

$X = \text{Sp}(A)$, $f \in A$, set $U = \{x \in X \mid |f(x)| < 1\}$.

$$U' = \{x \in X \mid |f(x)| > 1\}$$

$$U'' = \{x \in X \mid |f(x)| > 0\}$$

Then any finite union of sets of this type is admissible
any finite covering by finite unions of these sets is admissible.

Example: (less covering than canonical topology)

$D = \text{Sp}(k\langle T \rangle)$ closed unit disk, with $k = \bar{k}$, and residue field $\bar{k} := \mathcal{O}_{m_x}$ infinite

$D = \{x \in k : |x| \leq 1\}$ has a cover by admissible open $\bigsqcup_{[\alpha] \in \widehat{k}} D^-(\alpha, 1)$, that doesn't have a finite subcover, thus not admissible.

Proposition: Any sheaf on X with respect to the weak \mathcal{G} -topology admits a unique extension with respect to the strong \mathcal{G} -topology.

Prop (completeness) the strong \mathcal{G} -topology on a k -affinoid space X verifies:

(G0) \emptyset, X are admissible open

(G1) Let $U \subseteq X$ be an admissible open, $(U_i)_{i \in I}$ an admissible covering of U . Let $V \subseteq U$ be a subset such that $V \cap U_i$ is admissible, then V is admissible open in X .

(G2) Let $(U_i)_{i \in I}$ be a covering of an admissible open $U \subseteq X$, by admissible open $U_i \subseteq X$ such that it admits an admissible covering as a refinement. Then $(U_i)_{i \in I}$ is an admissible covering.

These completeness properties are the reason why we consider the strong \mathcal{G} -topology. They allow us to define a \mathcal{G} -topology from local pieces as the following proposition shows.

Prop (Gluing of complete \mathcal{G} -topology)

Let X be a set and $(X_i)_{i \in I}$ be a covering of X . Let \mathcal{J}_i be a \mathcal{G} -topology on X_i for every $i \in I$, such that the completeness is satisfied.

Assume $\forall i, j \in I, X_i \cap X_j \subseteq X_i$ is \mathcal{J}_i -open, and $\mathcal{J}_i|_{X_i \cap X_j} = \mathcal{J}_j|_{X_i \cap X_j}$, then there is a unique \mathcal{G} -topology \mathcal{J} on X such that:

① $\forall i \in I, X_i \subseteq X$ is \mathcal{J} -open and $\mathcal{J}|_{X_i} = \mathcal{J}_i$

② \mathcal{J} satisfies the completeness conditions (G0) (G1) and (G2)

③ $(X_i)_{i \in I}$ is a \mathcal{J} -covering of X .

\leadsto in particular, $U \subseteq X$ is admissible iff $\forall i, U \cap X_i$ is admissible.

$\{U_j\}$ of U is admissible iff $\forall i, \{U_j \cap X_i\}_j$ is admissible

Def: a **locally G -ringed k -space** is a pair (X, \mathcal{O}_X) where X is a G -topological space, \mathcal{O}_X is a sheaf of k -algebras on X with all the stalks $\mathcal{O}_{X,x}$ being local rings.

Prop: the functor

$$\left\{ \begin{array}{c} k\text{-affinoid spaces} \\ X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{locally } G\text{-ringed spaces} \\ (\text{strong } G\text{-topology, } \mathcal{O}_X) \end{array} \right\}$$

is fully faithful.

Def: the category of **rigid analytic k -spaces** has:

- objects: locally G -ringed k -space (X, \mathcal{O}_X) such that the G -topology satisfies the completeness condition.
- morphism: a morphism of rigid analytic k -spaces is a morphism of locally G -ringed space.

Prop: (**G -ring affinoid k -space**)

Given rigid analytic k -spaces $(X_i)_{i \in I}$, and admissible open $X_{ij} \subseteq X_i$, together with isomorphism $\varphi_{ij}: X_{ij} \rightarrow X_{ji} \forall i, j \in I$. Assume

- ① $\varphi_{ij} \circ \varphi_{ji} = \text{id}$, $X_{ii} = X_i$, $\varphi_{ii} = \text{id}_{X_i}$
- ② φ_{ij} induces isomorphisms $\varphi_{ijk}: X_{ij} \cap X_{ik} \rightarrow X_{ji} \cap X_{jk}$ s.t.
 $\varphi_{ijk} = \varphi_{kji} \circ \varphi_{ikj} \forall i, j, k$

Then X_i can be glued uniquely into a rigid k -space X , admitting (X_i) as an admissible covering.

Def: Let (Z, \mathcal{O}_Z) be a k -scheme of locally finite type. An **analytification** of (Z, \mathcal{O}_Z) is a rigid analytic k -space $(Z^{an}, \mathcal{O}^{an})$ with a morphism of locally G -ringed spaces

$$(t, t^*): (Z^{an}, \mathcal{O}^{an}) \longrightarrow (Z, \mathcal{O}_Z)$$

satisfying the **universal property**:

given a rigid analytic k -space (Y, \mathcal{O}_Y) and any morphism of locally G -ringed space $(Y, \mathcal{O}_Y) \rightarrow (Z, \mathcal{O}_Z)$, the latter factors through (t, t^*) via a unique morphism of rigid analytic k -spaces $(Y, \mathcal{O}_Y) \rightarrow (Z^{an}, \mathcal{O}^{an})$

$$\begin{array}{ccc} & & (Z^{an}, \mathcal{O}^{an}) \\ & \nearrow \exists! & \downarrow (t, t^*) \\ (Y, \mathcal{O}_Y) & \longrightarrow & (Z, \mathcal{O}_Z) \end{array}$$

Theorem (How to construct analytification)

By gluing property, it suffices to consider affine k -scheme of finite type. So we reduce to $\text{Spec}(k[\mathfrak{F}]/\mathfrak{a})$ for $\mathfrak{F} = (x_1, \dots, x_n)$ and $\mathfrak{a} \subseteq k[\mathfrak{F}]$ an ideal.

Let $c \in k$ and $|c| > 1$. Consider the closed n -dimensional polydisk of radius $|c|^{-1}$, with algebra of functions:

$$\overline{T}_n^{(i)} := \left\{ \sum a_\nu \mathfrak{F}^\nu \mid \lim_{|\nu| \rightarrow \infty} a_\nu \cdot |c|^{i|\nu|} = 0 \right\}$$

Then we have inclusions:

$$\overline{T}_n^{(0)} \hookrightarrow \overline{T}_n^{(1)} \hookrightarrow \overline{T}_n^{(2)} \dots \hookrightarrow k[\mathfrak{F}]$$

This induces:

$$\text{Sp}(\overline{T}_n^{(0)}/\mathfrak{a}) \hookrightarrow \text{Sp}(\overline{T}_n^{(1)}/\mathfrak{a}) \hookrightarrow \dots$$

Since the images of these inclusions are affinoid subdomains, and compatibility is also satisfied, we can glue $(\overline{T}_n^{(i)})_{i \in \mathbb{I}}$ to a rigid space, called

$$\text{Spec}(k[\mathfrak{F}]/\mathfrak{a})^{an} = \bigcup_{i=0}^{+\infty} \text{Sp}(\overline{T}_n^{(i)}/\mathfrak{a})$$

Corollary: the analytification map $Z^{an} \rightarrow Z$ identifies the points of Z^{an} as closed points of Z .

Pf: ① for $\mathbb{A}^{n,an}$, any point x in a closed ball of radius r , $m_x \subseteq k\langle r^{-1}T \rangle$, it's image is $\varphi^{-1}(m_x) = m_x \cap k[T]$. Then

$$k \subset k[T] / \varphi^{-1}(m_x) \longrightarrow k\langle r^{-1}T \rangle / m_x$$

By affinoïd Nullstellensatz, $k\langle r^{-1}T \rangle / m_x$ is finite over k thus $k[T] / \varphi^{-1}(m_x)$ is also a field, so m_x is maximal.

② On the other hand, \forall maximal ideal $m \subseteq k[T]$, $L = k[T]/m$ is a finite k . The norm extends uniquely to L . Choose r larger than any of the norm of coordinate. Then the evaluation map factors through $k\langle r^{-1}T \rangle$

$$\begin{array}{ccc} k[T] & \longrightarrow & k\langle r^{-1}T \rangle \\ ev \downarrow & \swarrow ev_r & \\ k[T]/m = L & & \end{array}$$

Then $m_x = ev_r^{-1}(0)$ is maximal and the image of m_x is m .

Example: assume k is algebraically closed.

① $\mathbb{A}^{n,an}$ is a union of nested sequence of closed unit disk, identified as closed points of $\mathbb{A}_k^n = k^n$

$$\mathcal{O}(\mathbb{A}^{n,an}) = \left\{ \sum c_i T^i \in k\langle T_1, \dots, T_n \rangle \mid \lim |c_i| r^{2i} = 0 \text{ for all } r > 0 \right\}$$

containing $k[T]$.

Admissible open and covering of $\mathbb{A}^{n,an}$ iff restricts to each $B(r)$ is admissible

② $\mathbb{P}^{n, \text{an}}$ is obtained by gluing $(n+1)$ copies of \mathbb{A}_k^n . But any point $x \in \mathbb{P}^n$ can be expressed in (x_0, \dots, x_n) such that $\max |x_i| = 1$, thus glue unit balls:

Let X_0, \dots, X_n be affinoid varieties isomorphic to \mathbb{B}^n , and

$$X_i = \text{Sp } k \left\langle \frac{z_0}{z_i}, \dots, \frac{z_n}{z_i} \right\rangle$$

where $\frac{z_i}{z_j}$ is a variable for $i \neq j$, and $\frac{z_i}{z_i} = 1$.

Define affinoid subdomains and gluing maps:

$$X_{ij} = X_i \left(\left(\frac{z_j}{z_i} \right)^{-1} \right) \subseteq X_i$$

$$\varphi_{ij}: X_{ij} \rightarrow X_j \text{ via}$$

$$k \left\langle \frac{z_0}{z_j}, \dots, \frac{z_n}{z_j}, \left(\frac{z_i}{z_j} \right)^{-1} \right\rangle \rightarrow k \left\langle \frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}, \left(\frac{z_j}{z_i} \right)^{-1} \right\rangle$$

$$\text{sending } \frac{z_m}{z_j} \mapsto \left(\frac{z_j}{z_i} \right)^{-1} \frac{z_m}{z_i}$$

③ analytic tori $\mathbb{A}^* / (q)$ see [BGR p.364]

Coherent sheaves and GAGA theorem

Let $X = \text{Sp} A$ be an affinoid k -space, M be an A -module. Consider the presheaf sending affinoid subdomains U to $M \otimes_A \mathcal{O}_X(U)$.

Similarly to Tate's acyclic theorem for \mathcal{O}_X , this presheaf $M \otimes_A \mathcal{O}_X$ is actually a sheaf.

Prop:

① The functor $\otimes_A \mathcal{O}_X : \text{Mod}_A \longrightarrow \text{Mod}_{\mathcal{O}_X}$ is fully faithful.
 $M \longmapsto M \otimes_A \mathcal{O}_X$

② A sequence of A -module $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact iff $0 \rightarrow M' \otimes_A \mathcal{O}_X \rightarrow M \otimes_A \mathcal{O}_X \rightarrow M'' \otimes_A \mathcal{O}_X \rightarrow 0$ is exact.

Def: Let X be a rigid analytic k -space, F an \mathcal{O}_X -module.

① F is called of **finite type** if \exists an admissible covering $(X_i)_{i \in I}$ of X and exact sequences

$$\mathcal{O}_X^{s_i} \Big|_{X_i} \rightarrow F \Big|_{X_i} \rightarrow 0$$

for $s_i \in \mathbb{N}^*$ and every $i \in I$.

② F is called **coherent** if it's of finite type and for any admissible open $U \subseteq X$, the kernel of any morphism of the form $\mathcal{O}_X^s \Big|_U \rightarrow F \Big|_U$ is of finite type.

Theorem (Kiehl's theorem)

Let $X = \text{Sp} A$ be an affinoid k -space, F be an \mathcal{O}_X -module. Then F is coherent iff F is associated to a finite A -mod.

Def: A morphism between affinoid spaces is a **closed immersion** if the morphism of affinoid algebra is surjective. A morphism between rigid analytic k -spaces is a **closed immersion** if there exists an admissible affinoid covering of the target, such that the pullbacks are a closed immersion of affinoid spaces.

Def: A morphism of rigid analytic k -spaces $\varphi: X \rightarrow Y$ is called **separated** if the diagonal $\Delta: X \rightarrow X \times X$ is a closed immersion.

Def: (relative compact) Let Y be an affinoid k -space, and X rigid k -space with a morphism $X \rightarrow Y$. Let $U \subseteq U' \subseteq X$ be affinoid subdomains. We say that U is **relatively compact** in U' , denoted as $U \Subset U'$ if there exists $r \in \mathbb{N}$ such that there exists a closed immersion $U' \hookrightarrow \mathbb{D}^r$ to the closed unit disk, inducing a monomorphism $U \hookrightarrow (\mathbb{D}^o)^r$ to the open unit disk.

Def: (proper morphism)

A morphism of rigid analytic k -spaces is called **proper** if

(1) φ is separated

(2) there exists an admissible covering $(Y_i)_{i \in I}$ of Y and for each $i \in I$, two finite affinoid covering $(X_{ij})_{j=1}^{n_i}$, $(X'_{ij})_{j=1}^{n_i}$ of $\varphi^{-1}(Y_i)$ such that $\forall i, j$, we have

$$X_{ij} \Subset_{Y_i} X'_{ij}$$

If Y is a point, X is proper means ① X is separated ② X admits an admissible covering that can be refined to an admissible covering relatively compact to the original one.

Lemma: $\varphi: X \rightarrow Y$ be a morphism of k -scheme of locally finite type.
 Then φ is proper if and only if $\varphi^{\text{an}}: X^{\text{an}} \rightarrow Y^{\text{an}}$ is proper.

Theorem (Kiel's proper mapping theorem)

Let $\varphi: X \rightarrow Y$ be proper morphism of rigid analytic k -space, let F be a coherent \mathcal{O}_X -module. Then $\forall q \geq 0$, $R^q \varphi_* F$ is coherent \mathcal{O}_Y -mod.

Base change via $X^{\text{an}} \rightarrow X$, we have an analytification functor

$$(\cdot)^{\text{an}}: \text{Mod } \mathcal{O}_X \longrightarrow \text{Mod } \mathcal{O}_{X^{\text{an}}}$$

Theorem (GAGA 1)

Let $f: X \rightarrow Y$ be a proper morphism of k -scheme of locally finite type
 For any coherent sheaf F on X , the canonical morphism

$$R^q f_* F \xrightarrow{\sim} R^q f_*^{\text{an}} F^{\text{an}}$$

is an isomorphism for all $q \geq 0$.

In particular, for X proper,

$$H^q(X, F) \xrightarrow{\sim} H^q(X^{\text{rig}}, F^{\text{rig}})$$

Theorem (GAGA 2)

Let X be a proper k -scheme. The analytification functor
 on \mathcal{O}_X -modules induces an equivalence of categories:

$$\text{Coh}(X) \longrightarrow \text{Coh}(X^{\text{an}})$$

In particular, \forall coherent $\mathcal{O}_{X^{\text{an}}}$ module F , $\exists!$ coherent \mathcal{O}_X -mod, s.t. $F = F^{\text{an}}$

\forall coherent \mathcal{O}_X -module F, G , we have $\text{Hom}_{\mathcal{O}_X}(F, G) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_{X^{\text{an}}}}(F^{\text{an}}, G^{\text{an}})$

Corollary: $X = \mathbb{P}^n$, $F' =$ coherent ideal of $\mathcal{O}_{\mathbb{P}^n, \text{an}} \rightsquigarrow$

Closed analytic subset of $\mathbb{P}^{n, \text{an}}$ are algebraic.