A mini-course on non-Archimedean geometry

1 / 2 Rigid analytic variety

3. Berkovich space and analytification

4. p-adic differential equations

Classical rigid geometry is a theory of analytic functions over fields k that are complete under non-Archimedean absolute value.

- every Cauchy sequence converges
- forgetting the Archimedean property:

$$\forall x, y \in \mathbb{R}_+, \exists n \in \mathbb{N} \text{ s.t. } \underbrace{x + \cdots + x}_{n} > y$$

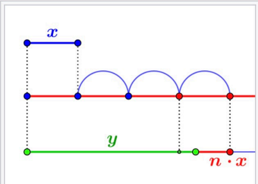


Illustration of the Archimedean property.

Such fields naturally appear in different areas.

Example of complete non-archimedean fields:

① $k = \mathbb{C}((t))$ consisting of $\sum_{i \geq n} c_i t^i$ with $n \in \mathbb{Z}$,

with absolute value defined by $\left| \sum_{i \geq n} c_i t^i \right| := e^{-n}$.

(complete)

② $\mathbb{Q}_p$ constructed by completing $\mathbb{Q}$ with p-adic absolute values

$|\cdot|_p$ defined as: $\left| p^r \frac{a}{b} \right|_p = p^{-r}$ for $a, b, r \in \mathbb{Z}$.

③ $\mathbb{C}_p$ by taking algebraic closure $\overline{\mathbb{Q}_p}^{alg}$ of $\mathbb{Q}_p$ and complete $\overline{\mathbb{Q}_p}^{alg}$ with respect to the unique extension of $|\cdot|_p$.

§0. Why study non-archimedean geometry?

§0.1. (number theory perspective)

there is a "local-global principle : a theorem holds over $\mathbb{Q}$ iff it holds over $\mathbb{R}$ and $\mathbb{Q}_p$ for all prime p". For example :

Theorem (Hasse-Minkowski)

Let $Q(x_1, \cdots, x_n)$ be quadratic form with $\mathbb{Q}$-coefficient. For $c \in \mathbb{Q}^*$, $Q(x_1 \cdots x_n) = c$ has a solution in $\mathbb{Q}$ iff in $\mathbb{R}$ and every $\mathbb{Q}_p$.

§0.2. (geometric perspective)

It's analogous to doing complex geometry.

Classically, give a lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\omega$ over $\mathbb{C}$ with $\mathrm{Im}(\omega) > 0$, $X := \mathbb{C}/\Lambda$ is a compact Riemann surface of $g=1$, topologically a torus.

It can be realized as solutions of polynomials as follows:

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \quad \text{meromorphic with poles at } \Lambda$$

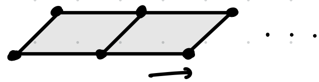
$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$$

Both $\wp(z)$ and $\wp'(z)$ are Λ-periodic, and descend to $\mathbb{C}/\Lambda$.

Consider the map
$$\begin{aligned} X &\xrightarrow{\cong} E_{\mathbb{C}} \subseteq \mathbb{P}^2_{\mathbb{C}} \quad \text{algebraic geometry.} \\ z &\longmapsto (\wp(z), \wp'(z), 1) \end{aligned}$$

where $E_{\mathbb{C}}$ is defined by $y^2 = 4x^3 - g_2 x - g_3$, for some $g_2, g_3 \in \mathbb{C}$.

A lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}w$ for $w \in \mathbb{C}$ can approach ∞ as $|nw| \to \infty$.



Now replace $\mathbb{C}$ by a non-archimedean field k. Say $k = \mathbb{Q}_p$.

But in non-archimedean world, $|nw|_p = |n|_p \cdot |w|_p \leq |w|_p$.

so this additive perspective of $\mathbb{C}$ fails.

Alternatively, consider $\mathbb{C}/\Lambda$ as two-step quotient:

$$\mathbb{C}/\Lambda \cong (\mathbb{C}/\mathbb{Z}) \big/ (\Lambda/\mathbb{Z}) \quad \text{as additive group}$$

① consider multiplicative structure on $\mathbb{C}$ via exponential map:

$$\exp : (\mathbb{C}/\mathbb{Z}, +) \xrightarrow{\cong} (\mathbb{C}^*, \times)$$
$$z \qquad\qquad \exp(2\pi i \cdot z)$$

mapping $\Lambda/\mathbb{Z}$ onto $q^{\mathbb{Z}}$, for $q = \exp(2\pi i w)$.

② $(\mathbb{C}/\mathbb{Z}) \big/ (\Lambda/\mathbb{Z}) \cong \mathbb{C}^*/q^{\mathbb{Z}} \rightsquigarrow k^*/q^{\mathbb{Z}}$ non-archimedean

since $\mathrm{Im}(w) > 0$, $0 < |q| = |\exp(2\pi i w)| < 1$ and as $n \to \infty$, $q^n \to 0$, $q^{-n} \to \infty$.

then $q^{\mathbb{Z}}$ is a discrete subgroup $\rightsquigarrow$ nice to do quotient.

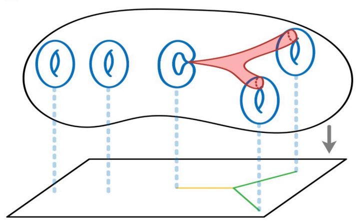
We will see k^* as a rigid analytic space and $q^{\mathbb{Z}}$ acts on k^*.

§ 0.3 Mirror symmetry and birational invariants

① non-archimedean SYZ conjecture:

Classical SYZ conjectures: near the "large complex structure limit", the Calabi-Yau manifold X_t admits a fibration $X_t \to B$, whose fibers are special Lagrangian tori.

The mirror dual $\check{X}$ is obtained by taking dual tori in each chamber of B and glue charts by wall-crossing.



Kontsevich-Soibelman proposed to view the family X_t as $t \to 0$, as a variety over non-archimedean field $\mathbb{C}((t))$.

The retraction from $\mathcal{X}^{an}$ to its skeleton is the SYZ fibration.

Count non-archimedean disks to obtain mirror.

(See Porta-Yu, 22)

② p-adic Gamma conjecture.

Consider quantum connection over p-adic field.

Recent works of [Bai-Pomerleano-Seidel, 25'] conjectured an overconvergent Frobenius structure and describes the asymptotics using p-adic Gamma function.

③ A-model decomposition and Hodge atom.

Gromov-Witten potential converges non-archimedeanly $\rightsquigarrow$ quantum connection can be decomposed analytically using rigid geometry, and gives rise to birational invariants.

[See Hinault-Yu-Zhang-Z. 24', Katzarkov-Kontsevich-Pantev-Yu 25']

§1. non-archimedean field

Let K be a field. A map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ is called non-archimedean absolute value if $\forall\, a, b \in K$, we have

① $|a| = 0 \iff a = 0$

② $|ab| = |a||b|$

③ $|a+b| \leq \max\{|a|, |b|\}$.

$\rightsquigarrow$ $|1| = 1$ and $|n \in \mathbb{N}| \leq 1$

It defines an non-archimedean valuation $v : K \to \mathbb{R} \cup \{\infty\}$, s.t.

① $v(a) = \infty \iff a = 0$

② $v(ab) = v(a) + v(b)$

③ $v(a+b) \geq \min\{v(a), v(b)\}$.

Lemma: in such a field, a series $\sum c_i$ is a Cauchy sequence iff $\lim\limits_{i\to\infty} |c_i| = 0$.
Thus if k is complete, $\sum c_i$ converges iff $\lim\limits_{i\to\infty} |c_i| = 0$.

Property of non-archimedean field.

① for any triangle in K, at least two sides are of equal length

② each point of a disk can serve as it center

③ if two disks intersect, then one contains the other

Prop: the topology on K by a non-archimedean norm is totally disconnected.

(any subset of more than one point is not connected)

proof: $\forall\, x \in K$, $D^-(x,r) = \{a \in K \mid |x-a| < r\}$ is open.

Its complement $C = \{a \in K \mid |x-a| \geq r\}$ is also open: $\forall\, y \in C$, $|y-a| \geq r$

consider the open set $D^-(y,r)$. If $z \in D^-(y,r)$, $|y-z| < r$.

$$|y-a| \leq \max\{|y-z|, |z-a|\}$$

Since $|y-a| \geq r$, $|y-z| < r$, this implies $|z-a| \geq r$ and $z \in C$.

So $C \supseteq D^-(x,r)$.

$\forall$ subset A of K, containing at least two points $x \neq y$, with $\delta = \frac{|x-y|}{2}$.

Then $A = (A \cap D^-(x,\delta)) \sqcup (A \cap D^-(x,\delta)^c)$, thus disconnected.

$\square$

* K may not be discrete. Say in $\mathbb{Q}_p$, $|p^n - 0| = p^{-n}$ so $\{p^n\}$ has the limit point 0 $\rightsquigarrow$ still possible to do calculus!

So classical Identity Theorem from complex analysis fails, and we can't do analytic continuation by "patching up" functions locally on k^n $\rightsquigarrow$

Need a good notion of "topology" to define a sheaf:

Method 1: change the "topology" (Tate's rigid analytic spaces)

Method 2: add more points to make the topology nice (Berkovich space)

Theorem. (ideal and zero locus correspondance)

Let A be an affinoid k-algebra and $Y \subseteq Sp\,A$ a subset.

① $V(I(Y)) = \overline{Y}^{zar}$ the Zariski closure of Y

② (Hilbert - Nullstellensatz)

$$I(V(a)) = \mathrm{rad}\, a$$

i.e. bijection between

$$\{\text{reduced ideals in } A\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{Zariski closed set of } Sp\,A\}$$

So far, we've been using the properties of T_n, A as a commutative algebra.

Topology obtained on T_n is similar to classical algebraic variety, i.e.

when $k = \bar{k}$, $\left\{\begin{array}{l} \bar{B}^n(\bar{k}) \\ \text{closed sets are } f_1 = \cdots = f_n = 0 \\ \text{for } f_i = \sum c_\alpha z^\alpha,\ \lim |c_\alpha| = 0 \end{array}\right.$ $\overset{\sim}{\longleftrightarrow}$ $Sp(T_n) = \left\{\begin{array}{l} \max(T_n) \\ \text{Zariski topology} \end{array}\right.$

$\rightsquigarrow$ too coarse that $\{x \in \bar{B}^n(\bar{k}) : |x| < \varepsilon\}$ isn't open, want to enlarge it.

$$\{x \in \bar{B}^n(\bar{k}) : |f(x)| < \varepsilon\} \subseteq \bar{B}^n(\bar{k}) \longleftrightarrow \{x \in Sp(T_n) : |f(x)| < \varepsilon\}$$

where $f(x)$ denotes $[f] \in T_n/m_x$

need to add more open sets to $\max(T_n)$ s.t. ε-ball is open.

Def: For any affinoid space $Sp\,A$, the canonical topology is the coarsest topology s.t. $\forall$ map $f \in A$, $Sp(A) \longrightarrow \mathbb{R}$ is open.
$$m_x \mapsto |f(x)|$$

Prop: the following are open & closed in the canonical topology:
$$\{ x \in Sp\,A : |f(x)| = \varepsilon \}$$
$$\{ x \in Sp\,A : |f(x)| \leq \varepsilon \}$$
for any $f \in A$ and $\varepsilon > 0$.

Viewing $\bar{B}^n(\bar{k})$ as $Sp(T_n)/Gal(\bar{k}/k)$, the topology is totally disconnected So we have too many open sets Instead, we define analytic functions on open subsets, where we can take limit, i.e. form a Banach algebra.

For this, we need to understand the norms on T_n.

Def: On T_n, there is a norm, called Gauss norm, defined by:
$$\left| \sum c_\alpha z^\alpha \right| = \max_\alpha |c_\alpha|.$$

Together with this norm $(T_n, |\cdot|)$ is a k-Banach algebra satisfying:

① $|f| = 0 \iff f = 0$
② $|cf| = |c||f|$
③ $|fg| = |f||g|$
④ $|f+g| \leq \max\{|f|, |g|\}$
⑤ T_n is complete under $|\cdot|$.

Theorem (maximal modulus principle)

$\forall f \in T_n$, and $x \in \bar{B}^n(\bar{k})$, $|f(x)| \leq |f|$ and $\exists$ a $x \in \bar{B}^n(\bar{k})$ s.t. $|f(x)| = |f|$.

Gauss norm on T_n induces a residue norm on affinoid algebra A for any $\alpha : T_n \twoheadrightarrow A$ given by :

$$|\alpha(f)|_\alpha := \inf_{a \in \ker\alpha} |f-a| = \text{infimum of all } f \in T_n \text{ representing } [f] \in T_n/a .$$

Prop : ① $(A, |\ |_\alpha)$ is a Banach algebra,

② the quotient map $\alpha : T_n \to T_n/a$ is open and continuous.

③ $\forall \bar{f} \in A$, $\exists f \in T_n$. s.t. $|\bar{f}|_\alpha = |f|$

We also have spectral semi-norm on A :

$$|f|_{sp} = \sup_{x \in Sp(A)} |f(x)|$$

Theorem (maximal modulus principle for A)

For any affinoid algebra A, and $f \in A$, $\exists x \in Sp(A)$ s.t. $|f(x)| = |f|_{sp}$.

It's a norm iff the intersaction of all maximal ideals in A is (0). Since A is Jacobson, this is equivalent to A being reduced. In this case, $(A, |\cdot|_{sp})$ is a Banach algebra.

Theorem (equivalent of norms)

Let $(A_1, |\cdot|)$, $(A_2, |\cdot|)$ be two k-affinoid algebras provided with norms for which they are Banach algebras. Let $\varphi : A_1 \to A_2$ be a homomorphism of k-algebras. Then u is continuous with respect to the given norms.

In particular, all norms on an affinoid algebra A, making it into a Banach k-algebra are equivalent.

So we don't need to keep track of norms used for an affinoid algebra, making it a Banach algebra. We want to define analytic functions on on open subsets, where we can take limit, i.e. form a Banach algebra. So naturally, we ask them to be affinoid algebra.

Def: Let $X = Sp(A)$ be an affinoid k-space, and $f_i, g_j \in A$. Consider the following open sets U, called special affinoid subdomains.

(1) $X(f_1, \cdots, f_n) = \{x \in Sp A : |f_i(x)| \leq 1\}$ called Weierstrass domain

(2) $X(f_1, \cdots, f_r, g_1^{-1}, \cdots, g_s^{-1}) = \{x \in Sp A : |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$ called Laurent domain

(3) $X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0}) = \{x \in Sp A \cdot |f_i(x)| \leq |f_0(x)|\}$ called rational domain

for $f_0, \cdots, f_r$ having no common zeros

<u>They satisfy the following universal properties:</u>

There exists a morphism of affinoid k-spaces $i: Sp(A) \to X$ s.t. $i(Sp(A)) \subseteq U$ and for any morphism of affinoid k-spaces $\varphi: Y \to X$, s.t. $\varphi(Y) \subseteq U$, $\exists$ a unique factorization $\forall \varphi$.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, dashed, "\exists !"'] & X \\ & Sp(A) \arrow[u, "i"'] \end{tikzcd}$$

view A as the ring of functions on U, even when $i(Sp(A)) \neq U$. Such U in general are called affinoid subdomains.

Now, to give the special affinoid subdomains an affinoid space structure, we define the relative Tate algebra over A by.

$$A\langle \xi_1, \cdots, \xi_r \rangle := \left\{ \sum_{J \in \mathbb{N}^r} a_J \xi^J . \ a_J \in A \text{ and } a_J \xrightarrow[|J| \to \infty]{} 0 \right\}$$

It has a norm $|\sum a_J \xi^J| = \max |a_J|$, making it a Banach algebra.

Proposition for these three special affinoid subdomains, the maps $i: Sp(A) \to X$ are bijective and is given by:

$$Sp\left(A\langle f \rangle = \frac{A\langle \xi_1, \cdots, \xi_r \rangle}{(\xi_i - f_i)} \right) \xrightarrow{\sim} X(f) \subseteq X \quad \text{Weierstrass domain}$$

$$Sp\left(A\langle f, g^{-1} \rangle = \frac{A\langle \xi_1, \cdots, \xi_r \rangle}{(\xi_i - f_i, 1 - g_j f_j)} \right) \xrightarrow{\sim} X(f, g^{-1}) \subseteq X \quad \text{Laurent domain}$$

$$Sp\left(A\langle \frac{f}{f_0} \rangle = \frac{A\langle \xi_1, \cdots, \xi_r \rangle}{(f_i - f_0 \xi_v)} \right) \xrightarrow{\sim} X\left(\frac{f}{f_0} \right) \subseteq X \quad \text{rational domain}$$

Prop (Properties of affinoid subdomains)

① (transitivity) for an affinoid space X, if $V \subseteq X$ and $U \subseteq V$ are affinoid subdomains, then $U \subseteq X$ is an affinoid subdomain.

② (pullback) if $X' \subseteq X$ is an affinoid subdomain, $Y \subseteq X$ a morphism of affinoid spaces then $X' \times_X Y \to Y$ is an affinoid subdomain.

③ (intersection) if $U \subseteq X$ and $V \subseteq X$ are affinoid subdomains, then $U \cap V \subseteq X$ is an affinoid subdomain.

④ each Weierstrass domain is a Laurent domain, and each Laurent domain is a rational domain.

⑤ given a rational domain $U \subseteq X$, $\exists$ a Laurent domain $U' \subseteq X$ such that $U \subseteq U'$ is a Weierstrass domain.

⑥ transitivity holds for Weierstrass, rational domains, not for Laurent domain

The following theorem shows that rational subdomains are like principle open subscheme in algebraic geometry.

Theorem (Gerritzen - Grauert)

Any affinoid subdomain is a finite union of rational subdomains

Cautious: union of affinoid subdomains, may not be affinoid. So they don't form a topology For example, take some $c \in k$, $0<|c|<1$,

In $T_2 = k\langle X.Y\rangle$, consider $U_1 \cup U_2$, where

$$U_1 = X(\tfrac{y}{c}) := \{ |y| \le |c| \}$$
$$U_2 = X(\tfrac{x}{c}) = \{ |x| \le |c| \}$$
$$U_1 \cap U_2 = X(\tfrac{x}{c}, \tfrac{y}{c}) := \{ |x| \le |c|, |y| \le |c| \}$$

By Tate acyclic theorem,

$$\begin{aligned} \mathcal{O}(U_1 \cup U_2) &= \mathrm{Ker}(\mathcal{O}(U_1) \oplus \mathcal{O}(U_2) \longrightarrow \mathcal{O}(U_1 \cap U_2)) \\ &= \{ \sum a_{ij} x^i y^j : \lim_{i+j \to \infty} |a_{ij}||c|^i = \lim_{m+n \to \infty} |a_{ij}||c|^j = 0 \} \\ &= \{ \sum a_{ij} x^i y^j : \lim |a_{ij}| = 0 \} \\ &= k\langle X.Y\rangle \end{aligned}$$

But $U_1 \cup U_2 \neq \overline{B}^2(\overline{k})$, it contradicts.