Introduction to Hodge atoms II

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"At a general point in the locus of Hodge classes, the generalized eigenspaces of $Eu*$ are compatible with Hodge structures, i.e. the generalized eigenspaces decomposes canonically into $\bar{\mathbb{Q}}$-linear representation of Mumford-Tate group Hod of $\mathbb{Z}/2$ graded polarizable pure Hodge struct."

[KKPY 25]

nc Hodge structures / F-bundles

Gromov-Witten invariants packaged in differential equations ⇝

(non-archimedean) decomposition according to eigenvalues of c_1* ⇝ Hodge atoms

motivic information ⇝

⇝ provide invariants under birational transformation.

⇝ contain information of Hodge structures

Hod($\bar{\mathbb{Q}}$)-representation

§1. Non-archimedean A-model F-bundles:

Let X be a smooth projective variety over $\mathbb{C}$ $(K = \bar{K} = \mathbb{C})$

Fix a field k, s.t. $\mathrm{char}(k) = 0$, and fix $\Bbbk$ an algebraically closed non-archimedean field $\Bbbk \supseteq k$, s.t. $v|_k$ trivial. Say $\Bbbk = \bar{k}((y^{\mathbb{Q}}))$.

Choose basis $\{T_i\}$ of $H^*(X;k)$, s.t. $T_0 = \mathrm{Id}$, $T_1, \cdots, T_m$ are algebraic classes $CH^{1,\mathrm{hom}}(X) \otimes k$ in $H^2(X;k)$, $T_{m+1} \cdots$ are basis of $H^2_{\mathrm{trans}} \oplus H^{\geq 3}(X)$

Gromov-Witten potential:

$$\Phi(q,t) = \sum_{\substack{n \geq 0 \\ \beta \in NE(X;\mathbb{Z})}} \frac{q^\beta}{n!} \sum_{i_1, \cdots, i_n} \langle T_{i_1}, \cdots, T_{i_n} \rangle t_{i_1} \cdots t_{i_n} \in k[\![NE]\!][\![t_0, \cdots, t_r]\!]$$

variables t_i are super variables with parity $\deg(T_i) \bmod 2$, and it lies in completed symmetric product of even variables tensored with exterior algebra in odd variables.

Q: What k-analytic space is $\Phi(q,t)$ analytic?

Construct an $\mathbb{k}$-analytic space for $\Phi(q,t)$

① for even degree cohomology, $B^{ev}_{X,t}$ defined as:

* $T_0 = 1$. By unit axiom, Φ is polynomial in t_0.

$\mathcal{O}(\mathbb{A}^{1,an}) = \{ \sum_{n\geq 0} a_n u^n \text{ s.t. } \forall r>0,\ |a_n| r^n \to 0 \} \supseteq$ polynomials

* $T_2, T_4, \cdots$ the coefficients are Gromov-Witten invariants $\in \mathbb{Q}$.

$\mathcal{O}(\text{open unit disk}) = \{ \sum_{n\geq 0} a_n t^n \text{ s.t. } \forall r<1,\ |a_n| r^n \to 0 \}$

② for odd degree cohomologies, $B^{odd}_{X,t}$ defined as:

$T_1, T_3, \cdots$ appear only as monomials since they are odd variables. Consider the super analytic variety, whose underlying analytic variety is a point, and algebra of functions is $\Lambda(H^{odd,\vee}_{(X;\mathbb{k})})$

③ for q parameter, define $B_{X,q}$ as follows:

Consider the torsion free part of Néron - Severi group

$$NS(X,\mathbb{Z})_{tf} := Im(cl: CH^1 \longrightarrow H^2(X;\mathbb{Z}))$$

For $\mathbb{G}_m = Spec(\mathbb{k}[T,T^{-1}])$, there exists valuation map from the Berkovich analytification

$$\begin{aligned} v: \mathbb{G}_m^{an} &\longrightarrow \mathbb{R} \\ x &\longmapsto -\log |T(x)| \end{aligned}$$

Now tensor with $NS(X,\mathbb{Z})_{tf} \otimes_{\mathbb{Z}}$, we have a continuous map

$$(v_1, \cdots, v_\rho): \left(NS(X,\mathbb{Z})_{tf} \otimes_{\mathbb{Z}} \mathbb{G}_{m,\mathbb{k}}\right)^{an} \longrightarrow NS(X;\mathbb{R})$$

Let $B_{X,q}$ be the open set $(v_1, \cdots, v_\rho)^{-1}(\text{Ample cone} \subseteq NS(X;\mathbb{R}))$

Lemma: Gromov-Witten potential Φ is a $\mathbb{R}$-analytic function on

$$B_X = B_{X.q} \times B_{X.t}^{ev} \times B_X^{odd}.$$

proof:

Choose ample line bundles $L_1, \cdots, L_n$ st $w_i = c_1(L_i)$ form a basis of $NS(X;\mathbb{Q}) = CH^{1,hom}(X) \otimes \mathbb{Q} \subseteq H^2(X)$.

Then the open simplicial cone

$$\sigma = \oplus \mathbb{R}_{>0} \cdot w_i$$

is in ample cone and form a cover by changing w_i and σ.

$B_{\sigma.q} = (v_1, \cdots, v_p)^{-1}(\sigma)$ form an open cover of $B_{X.q}$.

$$B_{\sigma.q} \cong \{(q_1, \cdots, q_n) : 0 < |q_i| < 1\}$$

Restricting $\Phi(t.q)$ to $B_{\sigma.q}$, we have

$$\Phi|_{B_{\sigma.q}} = \sum \frac{1}{n!} q_1^{(\beta \cdot w_1)} \cdots q_n^{(\beta \cdot w_n)} \left(\sum \langle T_{i_1} \cdots T_{i_n} \rangle_\beta t_{i_1} \cdots t_{i_n}\right)$$

Since $\beta \cdot w_i > 0$, $0 < |q_i| < 1$, with coefficient $\mathbb{Q}$ so it converges

this series converges $\square$.

Let $\mathbb{D}$ be the germ at 0 of an analytic disk with coordinate u.

$$\mathcal{O}(\mathbb{D}) = \{\sum a_n t^n : \text{for some } r > 0,\ |a_n| r^n \to 0\}.$$

Corollary: Denote by $\mathcal{H}$ the trivial vector bundle on $\mathcal{B}_X \times \mathbb{D}$ with fiber $H^\bullet(X;\mathbb{k})$. Then quantum product is analytic.

Define non-archimedean analytic quantum connection using the same formulas $(\mathcal{H}, \nabla)/B_X$. Over each chart $(B_6, \{q_j\}, \{t_i\})$, it is:

$$\nabla_{\partial u} = \partial u - \frac{Eu}{u^2} + \frac{Deg - \dim X \cdot id}{u}$$

$$\nabla_{\partial q_i} = \partial q_i - \frac{w_j *}{u\, q_i}$$

$$\nabla_{\partial t_i} = \partial t_i - \frac{T_i *}{u}$$

where Eu denotes the analytic Euler field s.t. $\forall\, \gamma \in B_6$

$$Eu_\gamma = c_1(T_X) + \frac{Deg - 2\, id}{2}(\gamma)$$

$\rightsquigarrow$ non-archimedean overmaximal F-bundle associated to smooth projective variety $X/_{\mathbb{C}}$, $\Bbbk \supseteq k$.

This is overmaximal, since ∂t_i and ∂q_i both correspond to H^2_{alg}.

Remove redundancy by restricting to the closed analytic set $\rightsquigarrow$ maximal.

$$H^2_{tran} \otimes_k \Bbbk \longrightarrow H^2 \otimes_k \Bbbk .$$

§.2 Non-archimedean F-bundles

Fix a non-archimedean field $\mathbb{k}$ of char 0, s.t. $v|_{\mathbb{Q}}$ trivial.

B a smooth $\mathbb{k}$-analytic space, $\mathbb{D}_u$ germ at 0 in a $\mathbb{k}$-analytic closed unit disk with coordinate u.

Define non-archimedean F-bundle similarly. Then for B an admissible open neighborhood of a rational point in a smooth $\mathbb{k}$-analytic space.

Spectral decomposition theorem and extension of framings theorem holds on an admissible open neighborhood of (b)

nc Hodge structures / F-bundles

Gromov-Witten invariants packaged in differential equations $\rightsquigarrow$

(non-archimedean) decomposition according to eigenvalues of $c_1\star$ $\rightsquigarrow$ Hodge atoms $\rightsquigarrow$ provide invariants under birational transformation.

motivic information $\rightsquigarrow$

Hodge atoms $\rightsquigarrow$ contain information of Hodge structures

$\mathrm{Hod}(\bar{\mathbb{Q}})$-representation

Special Mumford - Tate group.

Consider the embedding $s: \mathbb{C}^* \hookrightarrow GL_2(\mathbb{R})$ by

$$a+bi \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Deligne torus

the image is the $\mathbb{R}$-points of an algebraic subgroup $S \subseteq GL_2$, s.t.

$S(A)$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$, w. $a-d = b+c = 0$.

A $\mathbb{R}$-Hodge structure of $\mathbb{R}$-vector space $H_{\mathbb{R}}$, is a decomposition

$$H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p,q} H^{p,q} \qquad \text{s.t. } H^{p,q} = \overline{H^{q,p}}$$

It determines a representation $\rho: S(\mathbb{R}) \to GL(H_{\mathbb{R}})$ by $z = a+bi$ acts by $z^p \bar{z}^q$.

In particular, $i \in \mathbb{C}^*$ acts on $H^{p,q}$ by i^{p-q}

Lemma. let $\rho: S(\mathbb{R}) \to GL(H_{\mathbb{R}})$ be an algebraic representation, then ρ defines a $\mathbb{R}$-Hodge structure by $H^{p,q} = \bigcap_{a^2+b^2 \neq 0} \ker(\rho(s(a,b)) - (a+ib)^p (a-ib)^q)$

Def: Let H be a $\mathbb{Q}$-Hodge structure and $\rho: S(\mathbb{R}) \to GL(H_{\mathbb{R}})$ representation for $H_{\mathbb{R}}$.
The Mumford - Tate group of H is the smallest algebraic subgroup of $GL(H)$ over $\mathbb{Q}$ whose $\mathbb{R}$-points contain the image of ρ.

Example: $MT(\mathbb{Q}(n)) = \begin{cases} \mathbb{G}_m & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$

Prop: Let V be an rational subspace of $T^{\bar{d},\bar{e}}(H) := \bigoplus_{i=1}^{n} H^{\otimes d_i} \otimes (H^\vee)^{\otimes e_i}$ for $\bar{d}, \bar{e} \in \mathbb{N}^n$.
Then ① V is a sub Hodge structure iff stable under action of $MT(H)$
② $t \in H^{\bar{d},\bar{e}}(H)$ is a $(0,0)$ Hodge class iff it is invariant under $MT(H)$
③ In particular, the fixed locus consists of Hodge classes

- For $H^*(X;\mathbb{Q})$, we have $H^*(X;\mathbb{Q})^{MT} = \bigoplus_p (H^{2p}(X;\mathbb{Q}) \cap H^{p,p})$

Universal Mumford-Tate group.

every object has a dual.

A neutral Tannakian category over k is a rigid abelian tensor category $(C, \otimes)$ s.t. $\mathrm{End}(1) = k$ and $\exists$ an exact faithful k-linear tensor functor $\omega: C \to \mathrm{Vect}_k$.

Theorem:

For every such category $\mathcal{C}$, the following functor is represented by an affine group scheme, called Tannakian fundamental group

$$\mathrm{Aut}^{\otimes}.\ k\text{-algebras} \longrightarrow \text{groups}$$
$$R \longrightarrow \mathrm{Aut}(\omega_R)$$

Further, $\mathcal{C}$ is equivalent to $\mathrm{Rep}_k(G)$

(See Tannakian Categories, Deligne & Milne)

Example: $\mathcal{C}$ = polarizable pure $\mathbb{Q}$-Hodge structures

$\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ forgetful functor

the affine group scheme $\mathrm{Hod}_{\mathbb{Q}}^{pol}$ in this case is called universal Mumford-Tate group.

$\forall$ $\mathbb{Q}$-algebra R, $MT(R)$ consists of natural isomorphisms $\omega_R \Rightarrow \omega_R$, s.t.

① compatible with tensor product on $\mathcal{C}$

② compatible with unit $\mathbb{Q}(0)$. $\eta_{\mathbb{Q}(0)} = \mathrm{id}_R$

i.e. given morphisms $V_{\mathbb{Q}}^{p,q} \xrightarrow{f} W_{\mathbb{Q}}^{p,q}$ of $\mathbb{Q}$-Hodge structures, $MT(R)$ consists of

R-linear map η's :
$$\begin{tikzcd} V_R \arrow[r, "\eta"] \arrow[d, "f_R"'] & V_R \arrow[d, "f_R"] \\ W_R \arrow[r, "\eta'"'] & W_R \end{tikzcd}$$
s.t. $f_R \circ \eta = \eta \circ f_R$ and satisfy ①②

view as actions on R-modules $V_{\mathbb{R}} \otimes_{\mathbb{R}} R$ preserving natural morphisms of Hodge str, preserving ① tensor products and ② acting trivially on $R \otimes_{\mathbb{Q}} \mathbb{Q}(0) = R$

(1) For $\mathbb{Q}$-Hodge structure V of weight n, the action of $r \in R^\times$ on V by

$$\eta(v) = r^n \cdot v$$

is natural in morphism between Hdg str of weight n, compatible with tensor product and acts trivially on $\mathbb{Q}(0)$.

Since $\mathbb{G}_m(R) = R^\times$ for $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[t, t^{-1}])$, we have an

embedding $\iota : \mathbb{G}_m \hookrightarrow MT$.

denote by $\epsilon_{Hod} = \iota(-1) \in MT$, acting on weight n Hdg structure by $(-1)^n$.

(2) Given any $m \in MT(R)$, consider its action on $R \otimes \mathbb{Q}(-1) = R \otimes H^2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Q})$

$$R \longrightarrow R$$
$$1 \longmapsto m_{\mathbb{Q}(-1)}$$

By tensor compatibility, the action of m on $\mathbb{Q}(0) = \mathbb{Q}(-1) \otimes \mathbb{Q}(1)$ is

$m_{\mathbb{Q}(-1)} \cdot m_{\mathbb{Q}(1)} = m_{\mathbb{Q}(-1) \otimes \mathbb{Q}(1)} = m_{\mathbb{Q}(0)} = \mathrm{id} \in R$

So $m_{\mathbb{Q}(-1)} \in R^\times$, and we have a Lefschetz character

$$MT \longrightarrow \mathbb{G}_m$$

Its kernel is denoted as Hod, acting trivially on $\mathbb{Q}(-1) = H^2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Q})$, thus acting trivially on all Tate twist. (Call it Hodge group)

* $r \in R^\times$ acting by r^{p-q} on $V^{p,q}$ is an element of $Hod(R)$

* Relation with a MT of a Hodge structure:

For V a $\mathbb{Q}$-Hodge structure of pure weight k, $\langle V \rangle^{\otimes} \subseteq Hod_{\mathbb{Q}}^{pol}$ the subcategory.

Then $MT(V) = \mathrm{Aut}^{\otimes}(\omega|_{\langle V \rangle^{\otimes}})$, and we have a factorization from universal MT

$$MT \xrightarrow{\varphi_V} MT(V)$$

(By Tannakian formalism, $\langle V \rangle^{\otimes n} \subseteq Hod_{\mathbb{Q}}^{pol}$ is fully faithful $\Rightarrow \varphi_V$ is faithfully flat)

$\rightsquigarrow$ universal MT.

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§3. Motivic information on quantum product.

$\mathcal{C}^{\mathbb{C}}$: pure André motives of smooth projective $\mathbb{C}$-varieties

$H^{\bullet}_{B}(\ ; \mathbb{Q}) : \mathcal{C}^{\mathbb{C}} / \text{Tate motives} \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$ is a fiber functor.

Then $\mathrm{Mot}^{\mathbb{C}}$ are the corresponding Tannakian fundamental group, i.e.

$$\mathcal{C}^{\mathbb{C}} / \text{Tate motives} = \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Mot}^{K})$$

Prop: ① $H^2(X;\mathbb{Q})_{alg} = H^2(X;\mathbb{Q})^{\mathrm{Mot}^{\mathbb{C}}}$ invariant part.

② $\mathrm{Mot}^{\mathbb{C}}$ action on $H^2(X;\mathbb{Q})$ respects the decomposition

$$H^2(X) = H^2_{alg}(X) \oplus H^2_{tran}(X)$$

③ The supercommutative quantum product

$$H^*(X) * H^*(X) \longrightarrow H^*(X) \otimes \mathbb{Q}[\![NE]\!] \hat{\otimes}_{\mathbb{Q}} \widehat{\mathrm{Sym}}(H^{\bullet}_{B}(X))^{\vee}$$

is $\mathrm{Mot}^{\mathbb{C}}$-equivariant

↳ Gromov-Witten class are algebraic cycles

We have a functor

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Mot}^{\mathbb{C}}) \arrow[r, "h"] \arrow[d, equal] & \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Hod}) \\ \mathcal{C}^{\mathbb{C}} / \text{Tate motives} \arrow[ur, "H^{\bullet}_{B}(, \mathbb{Q}):"'] & \end{tikzcd}$$

in particular, Hod actions factor through $\mathrm{Mot}^{\mathbb{C}}$.

§4. Hodge atoms

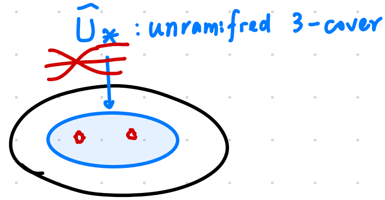
$K = \mathbb{C}$, $k = \mathbb{Q}$, $\mathbb{k} = \overline{\mathbb{Q}}((y^{a}))$

$\forall$ smooth projective variety X, $(H, \nabla)/_{B_X}$ is equivariant under $(\mathrm{Hod}(\mathbb{Q}), \epsilon_{\mathrm{Hod}})$

Consider the $\mathrm{Hod}(\mathbb{Q})$ fixed locus B_X^{Hod} in B_X, and $\tilde{B}_X^{\mathrm{Hod}}$ the ramified covering by eigenvalues of Eu action.

Let $U_X \subseteq B^{\mathrm{Hod}}$ the locus of maximal number of eigenvalues, and $\widehat{U}_X = \tilde{B}_{\mathrm{red}} \times_B U_X$

$\mathrm{Hod}(\mathbb{Q}) \curvearrowright B_X$



Def: The set of local atoms of X is the finite set $\pi_0(\widehat{U}_X)$ of connected components of $\widehat{U}_X$. The multiplicity of $\alpha \in \pi_0(\widehat{U}_X)$ is the degree of the cover $\widehat{U}_{X,\alpha} \longrightarrow \widehat{U}_X$.

Set of local Hod-atoms of sm projective varieties

$$\mathrm{Atom}^{K,\mathrm{loc}}_{\mathrm{Hod}} = \underset{[X]}{\bigsqcup} \pi_0(\widehat{U}_X) / \mathrm{Aut}(X).$$

isomorphism class

e.g.: Fano X over $\mathbb{C}$, say $\det(c_1 \star_q - x) = x^3(x^2 + q - 1)$.

$U_X = \mathbb{C}_q^* \setminus \{1\}$, $\widehat{U}_X = \{(q,x) \in \mathbb{C} \times \mathbb{C} \mid x^2 + q - 1 = 0\}$ connected Riemann-surface

then $x^3 \rightsquigarrow$ reduced structure 0, mul $= 1$

$x^2 - q - 1 \rightsquigarrow$ one connected component, mul $= 2$

Hodge atoms

① disjoint:

for $\mathcal{X}_1$, $\mathcal{X}_2$ smooth projective varieties, declare $[\alpha] \sim i[\alpha]$

where $i: \pi_0(\widehat{U}_{\mathcal{X}_1})/\mathrm{Aut}(\mathcal{X}_1) \hookrightarrow \pi_0(\widehat{U}_{\mathcal{X}_1 \sqcup \mathcal{X}_2})/\mathrm{Aut}(U_{\mathcal{X}_1} \sqcup U_{\mathcal{X}_2})$

② blowups:

X sm proj $\supseteq Z$ sm proj codim $m \geq 2$.

$\hat{X} = Bl_Z X$, $X' = X \sqcup Z \sqcup \cdots \sqcup Z$ $(m-1)$ disjoint union of Z.

Theorem (Iritani, and KKPY for non-archimedean)

$\exists$ non-empty connected open subsets $\hat{U} \subseteq B_{\hat{\mathcal{X}}}$, $U' \subseteq B_{\mathcal{X}'}$, and canonical isomorphism $({}^{\hat{X}}H, {}^{\hat{X}}\nabla)/_U \cong ({}^{X'}H, {}^{X'}\nabla)/_{U'}$

The subsets $\hat{U}_0 \subseteq \hat{U}$, $U_0' \subseteq \hat{U}$ of unramified spectral cover are connected

In induces a bijection of sets of connected component.

$$\pi_0(\widehat{U}_{\hat{\mathcal{X}}}) \cong \pi_0(\widehat{U}_{\mathcal{X}'})$$

Define $\alpha \in \pi_0(\hat{\mathcal{X}})/\mathrm{Aut}(\hat{\mathcal{X}}) \sim \alpha' \in \pi_0(\mathcal{X}')/\mathrm{Aut}(\mathcal{X}')$ iff related as above.

③ Similar for projective case.

Def. Hodge atom of smooth projective$/_{\mathbb{C}}$ is (local hodge atoms)$/_{\sim \text{ above}}$

§ 5. Chemical formula

For $\mathcal{X}/\mathbb{C}$, we call the following map the chemical formula of $\mathcal{X}$

$$CF_{Hod}(\mathcal{X}) = \sum_{\alpha \in \pi_0(\widetilde{U}_{\mathcal{X}})/Aut(\mathcal{X})} mul_{\mathcal{X}}(\alpha)\cdot \delta_\alpha : \quad \mathrm{Atoms}^{\mathbb{C}}_{Hod} \longrightarrow \mathbb{Z}_{\geq 0}$$

Prop: ① for $\mathcal{X}_i$ smooth projective,

$$CF(\mathcal{X}_1 \sqcup \mathcal{X}_2) = CF(\mathcal{X}_1) + CF(\mathcal{X}_2)$$

② for $\mathcal{X}$ smooth projective, $Z \subseteq X$ smooth projective of codim r

$$CF(Bl_Z \mathcal{X}) = CF(\mathcal{X}) + (r-1)\, CF(Z)$$

③ for $\mathcal{X}$ smooth projective, V rank r bundle over $\mathcal{X}$

$$CF(\mathbb{P}(V)) = r \cdot CF(\mathcal{X})$$

Prop: (Non-rationality criterion)
Suppose $\mathcal{X}$ smooth projective $\mathbb{C}$-variety dim $d \geq 2$. if $\mathcal{X}$ has a Hodge atom in the chemical formula, s.t. there doesn't exists smooth projective variety of dimension $\leq d-2$, that has α in its atomic composition, $\mathcal{X}$ isn't birational equivalent to $\mathbb{P}^d$.

proof: By weak factorization theorem, a birational map between $\mathcal{X}$ and $\mathbb{P}^d$ are given by blow up and blow down. Since atomic composition of $\mathbb{P}^d$, $CF(\mathbb{P}^d) = (d+1)\, CF(pt)$, we have $CF(\mathcal{X})$ consists only of atoms from varieties having dim $\leq d-2$.

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§6 Detect atoms using motivic info:

Proposition: We have a functor constructed as follows:

$$F : \mathrm{Atom}^{\mathbb{C}}_{\mathrm{Hod}} \longrightarrow \text{representation of } \mathrm{Hod}(\overline{\mathbb{Q}})$$

$[\alpha] =$  $\hat{b}$ $\qquad\qquad H^{\alpha}|_{b.u=0}$

① given $\alpha \in \mathrm{Atom}^{k,\mathrm{loc}}_{\mathrm{Hod}} = \coprod_{[\mathcal{X}]} \pi_0(\widetilde{U}_{\mathcal{X}}) / \mathrm{Aut}(\mathcal{X})$, associated to $\mathcal{X}$,

construct the maximal na A-model F-bundle $(\mathcal{H}, \nabla)/_{B_{\mathcal{X}} \times \mathbb{D}}$.

② Choose a rigid point b in $U_{\mathcal{X}}$

(b is in the fixed locus by Hod s.t. the spectral cover at b is unramified)

Choose one eigenvalue $\hat{b}$ over b in the component $[\alpha]$.

③ Consider generalized eigenspace decomposition of $\mathcal{H}_{b.u=0} = \oplus \mathcal{H}^{\lambda_i}_{b.u=0}$

By spectral decomposition theorem, $\mathcal{H}^{\lambda=b}_{b.u=0}$ extends to $(\mathcal{H}^{\hat{b}}, \nabla^{\hat{b}})/_{B^{\alpha} \times \mathbb{D}}$

④ Get a $\mathrm{Hod}(\mathbb{k})$ representation $\mathcal{H}^{\hat{b}}|_{b.u=0} = E^{\alpha}_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{k}$ for some $\mathbb{Q}$-vector space

Also a $\mathrm{Hod}(\overline{\mathbb{Q}})$ representation E^{α}.

Lemma: For Hod proreductive, $\overline{\mathbb{Q}}$ is algebraically closed, E^{α} is finite dim of $\mathrm{Hod}(\overline{\mathbb{Q}})$

the representation is controlled by discrete invariants and can't move continuously.

$\rightsquigarrow$ ⑤ $\mathrm{Hod}(\overline{\mathbb{Q}})$-rep E_{α} is independent of $\hat{b}$ chosen in $[\alpha]$.

We have the following invariants of atoms α:

① $\dim(\mathcal{H}^{\alpha}|_{b.u=0})^{\mathrm{Hod}(\overline{\mathbb{Q}})}$

② Hodge polynomial $P_{\alpha} \in \mathbb{Z}[t^{\pm}]$, s.t. coefficient of t^k is dimension of $\mathcal{H}^{\alpha}|_{b.u=0}$

with $(p-q)$ degree equal to k.

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§.7 Applications to rationality problem

Theorem 6.8. a very general 4-dim cubic hypersurface in $\mathbb{CP}^5$ isn't rational.

Pf: Assume rational Hodge classes are powers of $P = c_1(\mathcal{O}_{\mathbb{P}^5}(1))|_X$ for very general cubic hypersurface.

Use $P = c_1(\mathcal{O}(1))$ to project $\mathbb{C}[NE] \longrightarrow \mathbb{C}[q]$ $\dim H = 27$.

Lemma: suppose (G, ϵ_G) is an algebraic reductive group over k. $(A, \circ, 1_A)$ is a finite dimensional commutative unital $\bar{k}$-superalgebra ϵ_G acts by parity. If $a \in A^G$, then $a: A \longrightarrow A$, $a|_{A^G}: A^G \longrightarrow A^G$ and $a|_{A^{ev}}: A^{ev} \longrightarrow A^{ev}$ have the same reduced spectrum.

Consider the subbundle spanned by $h = \{1, P, P \cup P, \cdots\} \subseteq H^*(X; \mathbb{C})$.

Theorem: $P \underset{q}{*}$ actions preserves $\mathbb{C}[q]$-module $h \otimes_{\mathbb{C}} \mathbb{C}[q]$.

Pf: By deformation invariants or Coates-Givental,

$\langle c_1^m, c_1^n, T \rangle_{0,3,d} = 0$ for T non-ambient classes.

Then the reduced spectrum of $c_1 \underset{q=1}{*}: H \longrightarrow H$, can be computed by $c_1 \underset{q=1}{*}: h|_{q=1} \longrightarrow h|_{q=1}$.

For four-dim cubic, it follows from Givental's computaton that

$$K = 3\begin{pmatrix} 0 & 0 & 6q & 0 & 0 \\ 1 & 0 & 0 & 15q & 0 \\ 0 & 1 & 0 & 0 & 6q \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$p(\lambda) = 3^5(\lambda^5 - 3^6\lambda^2)$

reduced spectrum $= \{0, 9, 9\mathfrak{z}, \mathfrak{z}^2\}$.

So $(H.\nabla)/B_X^{Hod}$ decomposes as $\displaystyle\boxplus_{\lambda\in\{0,9,9\mathfrak{z},9\mathfrak{z}^2\}} (H^\lambda, \nabla^\lambda)/U_\lambda$

Consider the following invariants of atom α, from Hod-repentation $\mathcal{H}^\alpha|_{b,u=0}$

① $\forall$ Hodge atom α of X, $\rho_\alpha := \dim_{\bar{\mathbb{Q}}}(E^\alpha)^{Hod(\bar{\mathbb{Q}})} \leq \max(\dim H^\lambda_{b,0}) = 2$

② dimension of subspace in $\mathcal{H}^\alpha|_{b,u=0}$ with $(p-q)$ degree $=2$, equals 1,

↑ since $h^{3,1}(X) = 1$.

Such Hodge atom α can't appear pts, curves and surfaces

① For Hodge atom, from point or curves, $\mathrm{Coeff}_{t^2}(P_\alpha) = 0$

② For Hodge atom, from surface with $h^{2,0} = 0$, $\mathrm{Coeff}_{t^2}(P_\alpha) = 0$

③ For Hodge atom, from surface X with $h^{2,0} \neq 0$:

$$1 = \mathrm{Coeff}_{t^2}(P_\alpha) = (\text{subspace of } E^\alpha \subseteq H \text{ s.t. } \bar{\mathbb{Q}}^\times \subseteq Hod(\bar{\mathbb{Q}}) \text{ acts by weight } 2)$$
$$\leq \dim H^{2,0}(S)$$

Such surface is nef, and has only one atom, with $E^\alpha = H_B^*(S, \bar{\mathbb{Q}})$.

Since S projective, it has an algebraic cycle of dim 2, and H^0, H^4.

$\Rightarrow \rho_{\eta(S)} \geq 3$, contradicts □

nc Hodge structures / F-bundles

Gromov - Witten invariants packaged in differential equations $\rightsquigarrow$

(non-archimedean) decomposition according to eigenvalues of $c_1 *$ $\rightsquigarrow$ Hodge atoms $\rightsquigarrow$ provide invariants under birational transformation.

motivic information $\rightsquigarrow$

$\rightsquigarrow$ contain information of Hodge structures

$\mathrm{Hod}(\bar{\mathbb{Q}})$ -representation

Theorem: X sm proj CY, the Hodge numbers of X can be reconstructed from the atomic F-bundle of X.

nc Hodge structures / F-bundles

Gromov - Witten invariants
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