Introduction to nc-Hodge structures and F-bundles.

Shaowu Zhang Caltech.

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Reference:

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Quantum cohomology of projective bundles, Iritani-Koto, 2023

Topological Laplace transform and decomposition of nc-hodge structures, Yu-Zhang, 2024

Decomposition and framing of F-bundles and applications to quantum cohomology, Hinault-Yu-Zhang-Zhang, 2024

Birational invariants from Hodge structures and quantum multiplications, Katzarkov-Kontsevich-Pantev-Yu, 2025

nc Hodge structures / F-bundles

Gromov - Witten invariants
packaged in differential equations

(non - archimedean) decomposition
according to eigenvalues of $c_1 *$

motivic information

Hodge atoms

provide invariants under
birational transformation.

contain information of Hodge
structures

Hod($\bar{\mathbb{Q}}$)-representation

§1. nc-Hodge structure and vanishing cycle decomposition.

* analytic version of F-bundle
* arises naturally in Hodge theoretic aspect of mirror symmetry

A - model	B - model
curve counting	singularity theory
X : Smooth Fano variety / $\mathbb{C}$	(Y, f) : Y smooth quasi-proj / $\mathbb{C}$ and $f: Y \to \mathbb{A}^1$ proper

* information can be packaged into differential equations with a rational structure on solutions.

Def: an nc-Hodge structure consists of

① de Rham data:

(H, ∇) an algebraic vector bundle on $\mathbb{A}^1_u$, and ∇ has an irregular singularity at 0 and regular at ∞.

② Compatible Betti data : more later

A model example

N_d the number of rational curves in $\mathbb{P}^2$ of degree d passing through $3d-1$ points in general position.

$N_d = 1, 1, 12, 620, 87304, 26312976, \cdots$

$\rightsquigarrow$ example of Gromov-Witten invariants

Kontsevich recursive formula:

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1\geq 1, d_2\geq 1}} N_{d_1} N_{d_2} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right)$$

In general, GW invariants satisfy WDVV equation for smooth projective X

Quantum product:

$H = H^*(X;\mathbb{C})$ with basis $\{T_1, \cdots, T_n\}$, choose an ample class w.

If X Fano, then product is defined over $\mathbb{C}[q]$

$$H \otimes H \longrightarrow H \otimes \mathbb{C}[q]$$

$$T_i * T_j \longmapsto \sum_k \sum_{\substack{\beta \in NE(X) \\ \beta \cdot w = d}} \left(\sum_{n\geq 0} \frac{q^d}{n!} \langle T_i T_j T_k \rangle_{0,3}^{\beta} \right) T_k$$

Quantum connection. $(H, \nabla) / \mathbb{C}[u]$ restricted to $q = q_0$:

$$\nabla_{\partial u} = \partial u - \frac{c_1 *_{q_0}}{u^2} + \frac{Gr}{u}$$

Theorem (Chen 2024, Pomerleano-Seidel 2023 with added assumptions)

X monotone symplectic, $QDM(X)$ is of exponential type,

i.e. $\exists\, c_1, \cdots, c_n \in \mathbb{C}$ s.t. $QDM(X) \otimes \mathbb{C}((u)) = \bigoplus_{c_i} e^{c_i/u} \otimes (R_i, \nabla_i)$

where c_i are eigenvalues of $c_1 *$

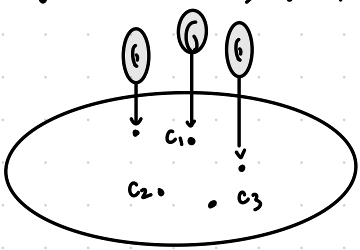
(R_i, ∇_i) is regular singular at 0

$e^{c_i/u} = (\mathbb{C}[u], d - d(\frac{c_i}{u}))$

proof via characteristic p method / Fourier-transform

B model example:

The mirror to Fano X is a Landau-Ginzburg model:
Y quasi-proj smooth / $\mathbb{C}$, $f : Y \to \mathbb{A}^1$ proper



Theorem (Sabbah, Kontsevich-Baranikov)

Twisted de Rham cohomology $H = \mathbb{H}^{\bullet}(Y, (\Omega^{\bullet}_Y[u], ud - df))$ is a free $\mathbb{C}[u]$ module. The natural connection $\nabla_{\partial u} = \partial u + f/u^2$ is of exponential type.

Proof via Fourier-Laplace transform and D-module theory.

* Exponential type singularity is the simplist among irregular singularity of differential equation, and can be well-understood via Fourier-Laplace transform of $\mathbb{C}[u]\langle\partial u\rangle$ module.

$$\begin{aligned} \mathbb{C}[u]\langle \partial u \rangle &\longrightarrow \mathbb{C}[\xi]\langle \partial \xi \rangle \\ u &\longmapsto -\partial \xi \\ \partial u &\longmapsto \xi \end{aligned}$$

Lemma: a $\mathbb{C}[u]\langle\partial u\rangle$ module M is regular holonomic everywhere iff as $\mathbb{C}[\xi]\langle\partial\xi\rangle$ module, it's exponential type at ∞ and regular at 0.

* Stationary phase formula relates vanishing cycles of M at c_i with (R_i, ∇_i) in the decomposition $e^{c_i/u} \otimes (R_i, \nabla_i)$, on the level of D-module.

Understanding exponential type in terms of its solutions.

Theorem: (analytic: uniqueness and existence)
Let D be a domain in $\mathbb{C}$. Consider the following differential eq.

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\cdot y = 0$$

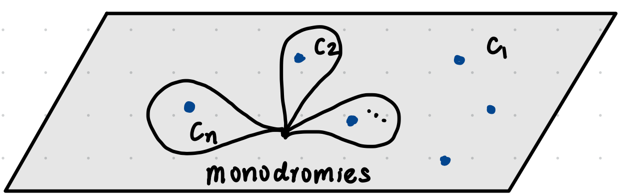
where $P_n(x)$ are holomorphic functions on D
Then the set of holomorphic solutions on the universal cover $\tilde{D}$ has $\dim_{\mathbb{C}} = n$. In particular, if D is a disk, $\dim_{\mathbb{C}}(\text{solutions on } D) = n$.

Def. we say $a \in \mathbb{C}$ is regular singular point for

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\cdot y = 0$$

if $P_i(x)$ are meromorphic with a pole of order $\leq i$ at a.
irregular singular otherwise.

When a differential equation has regular singularities, then the dimension of solutions around a singular point drops. But they form a constructible sheaf on $\mathbb{C}$.



Theorem (Levelt 75': formal solutions)
If 0 is an irregular singularity, then $\exists$ n linearly independent solutions of the form

$$y = e^{Q(x)} \cdot x^{\lambda} \cdot (\varphi_1 + \varphi_2 \ln x + \cdots + \varphi_s \cdot (\ln x)^s)$$

where $Q(x)$ is a polynomial in $x^{-1/q}$ for $q \in \mathbb{N}_+$, $\lambda \in \mathbb{C}$, $s \in \mathbb{N}$

Def. $\underline{P} \in \mathbb{C}[x]\langle\partial\rangle$ is of *exp type* if all its solutions are linear combination of

$$q = 1 \quad \text{and} \quad Q(x) = -\frac{c_i}{x} \text{ for some } c_i \in \mathbb{C}$$

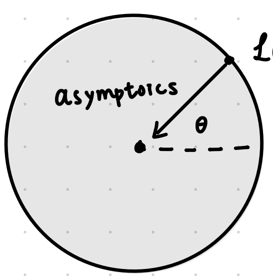
Example: $x^2\partial + (x+1)$ Solution: $\frac{1}{x} e^{\frac{1}{x}}$

For an irregular singularity, formal solution + asymptotic growth in terms of Stokes structures classify differential equations. It's called irregular Riemann-Hilbert correspondence:

Theorem (see Deligne-Malgrange, §3)
We have an equivalence of category

$$\left\{ \begin{array}{l} (H, \nabla) \text{ algebraic connection on } \mathbb{A}^1 \setminus 0 \\ \quad \text{exp type singularity at } 0 \\ \quad \text{regular at } \infty \end{array} \right\} \xrightarrow{RH} \mathbb{C}\text{-co-Stokes structure of exp } (\mathcal{L}, \mathcal{L}_{<\xi})$$

sending $(H, \nabla) \longmapsto$ $\mathcal{L}_\theta$: solution of ∇ around 0
$\mathcal{L}_{<\xi,\theta}$: Solution asymptotic to $\sum e^{\frac{c_i}{u}} \cdot \hat{f}$ with $\mathrm{Re}(c_i e^{-i\theta}) < \mathrm{Re}(\xi e^{-i\theta})$



$\mathcal{L}_\theta \cong \mathcal{L}_{<\xi,\theta}$

Def: A k-coStokes structure of exponential type $(\mathcal{L}, \mathcal{L}_<)$ with exponents $c_1, \cdots, c_n \in \mathbb{C}$ consists of

- a k-local system $\mathcal{L}$ on S^1 of finite dim fiber.
- a family of subsheaves $\mathcal{L}_{<\xi} \subseteq \mathcal{L}$ for $\xi \in \mathbb{C}$ s.t.

① $\forall\, \theta$, $\xi_1 \underset{\theta}{\leq} \xi_2$ (i.e. $\mathrm{Re}(\xi_1 e^{-i\theta}) < \mathrm{Re}(\xi_2 e^{-i\theta})$) implies $\mathcal{L}_{<\xi_1,\theta} \subseteq \mathcal{L}_{<\xi_2,\theta}$

② $\exists$ local systems $gr_{c_1}\mathcal{L}, \cdots, gr_{c_n}\mathcal{L}$ on S^1 s.t. locally $(\mathcal{L}, \mathcal{L}_<)$ is a direct sum of them, compatible with filtrations.

Def: an nc-Hodge structure of exp type consists of

① de Rham data:

(H, ∇) an algebraic vector bundle on $\mathbb{A}^1_u$, and ∇ has an irregular singularity at 0 of exponential type and regular at ∞.

② Betti data:

$(\mathcal{L}, \mathcal{L})$ a $\mathbb{Q}$-co-Stokes structure of exponential type.

s.t. they satisfy the following axioms:

* ($\mathbb{Q}$-struiture axiom)

(H, ∇) is a compatible $\mathbb{Q}$-co-Stokes structure $(\mathcal{L}, \mathcal{L}_<)$

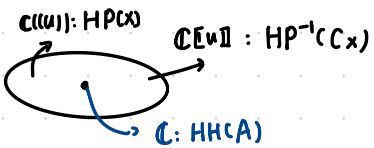
i.e. $\exists$ an isomorphism iso: $(\mathcal{L}, \mathcal{L}_<) \underset{\mathbb{Q}}{\otimes} \mathbb{C} \xrightarrow{\cong} RH(H, \nabla)$

* (opposedness axiom) "encoding Hodge decomposition"

Example: (Why nc Hodge? from nc geometry)

$C_X = D(\mathbb{Q}Coh(X))$

Assume affine $ncSpec(A)$



$\begin{cases} \bullet \text{ degeneration conj. v.b.} \\ \bullet\ \nabla: \text{Getzler-GM} \\ \bullet\ \mathcal{E}_B: \text{K theory} \end{cases}$

Example of A-model: quantum connection of Fano (H, ∇)

Gamma conjectures: $(\mathcal{L}, \mathcal{L}_<)_{\mathbb{Q}}$

§2. Decomposition of nc Hodge structure via Fourier-Laplace transform

↳ (H,∇) and $(\mathcal{L},\mathcal{L}_<)$

$$
\begin{tikzcd}
(H,\nabla) \text{ exp type} \arrow[rrr, "\text{Fourier - Laplace}"] \arrow[d, "\text{solution}"', "\text{irregular}"] & & & \text{regular holonomic } M \quad DR(M)=0. \arrow[d, "\text{regular.}"] \\
(\mathcal{L},\mathcal{L}_<)\otimes\mathbb{C} & & & \mathcal{F}\otimes\mathbb{C} \\
(\mathcal{L},\mathcal{L}_<) \arrow[u] \arrow[rrr, dotted, leftrightarrow, "?", "\text{Topological Laplace transform}"'] & & & \mathcal{F} \arrow[u]
\end{tikzcd}
$$

Theorem: (Malgrange 91, KKP 08, Sabbah 12)

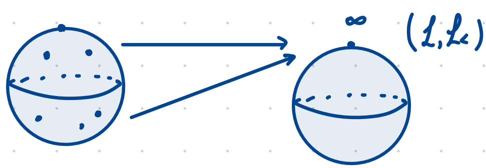
The direction from regular $\mathcal{F}_k$ to irregular $(\mathcal{L},\mathcal{L}_<)_k$ at ∞ is given by:

Φ: Constructible sheaf on $\mathbb{C}$ with $H^* = 0$ $\longrightarrow$ $Sto_<$ of exp type.

$$\mathcal{F} \longmapsto \begin{array}{l} \mathcal{L}_\theta := \mathcal{F}(\overline{H_{\infty,\theta}}) \\ \mathcal{L}_{<\xi,\theta} := \mathcal{F}(\overline{H_{Re(\xi\cdot e^{-i\theta}),\theta}}) \end{array}$$

It is compatible with the Riemann-Hilbert correspondence

* More generally, Mochizuki study the case corresponding to holonomic D-modules and describes the Stokes structures at ∞.



Q: How to obtain constructible sheaf $\mathcal{F}$ on $\mathbb{C}$ from $(\mathcal{L},\mathcal{L}_<)$ at ∞?

Theorem (Yu-Zhang 24, see Sabbah 25 for D-mod perspectives)

The direction from irregular $(\mathcal{L},\mathcal{L}_<)$ to regular $\mathcal{F}$ is :

$$\begin{array}{ccc} \text{Sto of exp type} & \longrightarrow & \text{Constructible sheaf on } \mathbb{C} \text{ with } H^*=0 \\ (\mathcal{L},\mathcal{L}_<) & \longmapsto & \forall\, c\in\mathbb{C},\ \ \mathcal{F}_c := H^1(S^1,\mathcal{L}_{<c}) \end{array}$$

proof :

Step 1: Assemble co-Stokes structure $(\mathcal{L},\mathcal{L}_<)$ to a sheaf $\mathcal{G}$ on $\mathbb{C}^*$.

- $\forall$ index $\xi=\lambda\cdot e^{i\theta}\in\mathbb{C}$ of $\mathcal{L}_{<\xi}$, consider an embedding

$$\begin{array}{ccccc} S^1 & \longrightarrow & S_\xi & \subseteq & \mathbb{C}^* \\ x & \longmapsto & r\cdot e^{ix}, & & \text{where } r=e^{\mathrm{Re}(\xi\cdot e^{-ix})} \end{array}$$

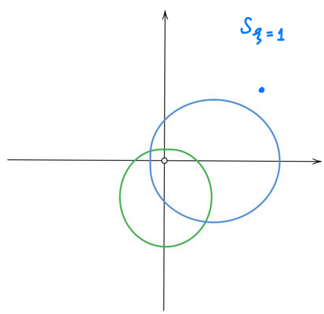


FIGURE 1. Examples of $S_\xi\subset\mathbb{C}^*$.

Define a $\mathcal{G}$ whose stalk at z having argument θ, and on circle S_ξ is

$$\mathcal{G}_z := \mathcal{L}_{<\xi,\theta}.$$

Step 2: Construct $\mathcal{F}$ from $\mathcal{G}$:

$$\begin{tikzcd} & \mathbb{C}^*\times\mathbb{C} & \\ & \bigsqcup S_\xi\times\{\xi\} \arrow[u, hook] \arrow[dl, "q_2"'] \arrow[dr, "p_2"] & \\ \mathcal{G} \text{ on } \mathbb{C}^* & & \mathbb{C} \end{tikzcd}$$

$\mathcal{F} := p_{2!}\, q_2^{!}\, \mathcal{G}$, so $\mathcal{F}_\xi = H^1(S^1,\mathcal{L}_{<\xi})$

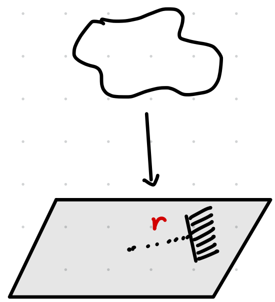
Application 1: B model $\mathbb{Q}$-structure

Prop (Sabbah 12', Yu-Z, 24') [$\mathbb{Q}$-structure axiom is also satisfied]

Let Y be a smooth quasi-projective variety$/_{\mathbb{C}}$, $f: Y \to \mathbb{A}^1$ proper map

De Rham data

$$\begin{cases} H = \mathbb{H}^\bullet(Y, (\Omega^\bullet_Y[u], ud - df)) \\ \nabla_{\partial_u} = \partial_u + \dfrac{f}{u^2} \end{cases}$$



has a compatible Betti data, s.t. $RH(H, \nabla) = (\mathcal{L}, \mathcal{L}_<) \otimes_{\mathbb{Q}} \mathbb{C}$

$$\begin{cases} \mathcal{L}_{\mathbb{Q},\theta} = H^\bullet(Y, f^{-1}(H_{\infty,\theta}); \mathbb{Q}) \\ \mathcal{L}_{\mathbb{Q},\le r,\theta} = H^\bullet(Y, f^{-1}(H_{r,\theta}); \mathbb{Q}) \end{cases}$$

proof:

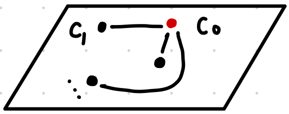
$$\begin{tikzcd} (H, \nabla) \arrow[r, "\text{Fourier-Laplace}"] \arrow[d, dotted, dash, "?"] & H^q f_* \mathcal{O}_X \arrow[d, leftrightarrow, "\text{D-module theory.}"] \\ (\mathcal{L}, \mathcal{L}_<) \arrow[r, "\Phi"] & H^{q-1} \Pi (Rf_* \mathbb{C}[\dim Y]) \end{tikzcd}$$

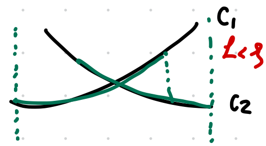
where Π is the projector $D^b_c(\mathbb{A}^1) \to D^b_c(\mathbb{A}^1)$.

Application 2: Relate spectral decomposition & vanishing cycle decomposition.

Let $(\mathcal{L}, \mathcal{L}_<)$ with exponents $\{c_1, \cdots, c_n\}$, and $\mathcal{F} = \Phi(\mathcal{L}, \mathcal{L}_<)$ over any field k.
Let θ not an anti-Stokes-direction

① (vanishing cycle decomposition of $\mathcal{F}$)
Choose c_0 near infinity with angle θ, and lines c_i to c_0





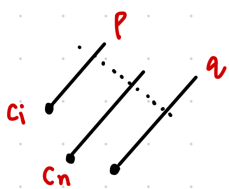
we have a decomposition $\mathcal{F}_{c_0} \xrightarrow{\cong} \bigoplus_i \mathcal{F}_{c_0} / \mathcal{F}_{c_i}$

② (Stokes decomposition of $\mathcal{F}$ from $(\mathcal{L}, \mathcal{L}_<)$)
We have a unique trivialization of $(\mathcal{L}, \mathcal{L}_<)$ over $I_\theta = (-\frac{\pi}{2}+\theta, \frac{\pi}{2}+\theta)$

$$(\mathcal{L}, \mathcal{L}_{<\xi})|_{I_\theta} = \bigoplus \beta_{c_i<\xi,!}\, \beta^*_{c_i<\xi}\, gr_{c_i}\mathcal{L}$$

Via topological Laplace transform, this induces $\forall c_0 \in \mathbb{C}$,
a decomposition $\mathcal{F}_{c_0} = H^1(S^1, \mathcal{L}_{<c_0}) = \bigoplus H^1(S^1, \beta_{c_i<\xi,!}\, \beta^*_{c_i<\xi}\, gr_{c_i}\mathcal{L})$

Theorem (Yu-Z, 24') $\forall \theta$: non-anti-Stokes direction, Stokes decomposition and vanishing cycle decomposition by straight lines of $\mathcal{F}$ agree at ∞_θ



We further show that the $\mathbb{Q}$-Stokes decomposition of $\mathcal{F}_{\xi,\mathbb{Q}}$ by straight lines is isomorphic to asymptotic lift of vanishing cycle decompositions of (H, ∇).

Theorem: for an nc-Hodge structure of exponential type,
the asymptotic lift of spectral decomposition along θ
= vanishing cycle decomposition by straight lines at ∞_θ

Now, allow variations of nc Hodge structures $\longrightarrow$ F-bundles.

§3. Logarithmic F-bundle and framing

$B = \mathrm{Spf}\, k[[t_1, \dots, t_n, q_1, \dots, q_n]]$, $D = V(q_1 q_2 \cdots q_n)$ normal crossing divisor

Def: A Logarithmic F-bundle over (B, D) is a vector bundle $\mathcal{H}$ on $B \times \mathrm{Spf}(k[[u]])$
∇ is a meromorphic flat on $\mathcal{H}$ with poles in $u=0$, and
$\nabla_{u^2\partial_u}$, $\nabla_{u\partial t_i}$, $\nabla_{u q_j \partial q_j}$ regular

Example 1: (logarithmic A model F-bundle)

Fix a basis $(T_i)_{0 \le i \le N}$ for $H^*(X;k)$, $\Delta(a) \in H^*$ where $\underset{\Delta(a)}{*}$ is well-defined.

Let $U = \mathrm{Spf}\, k[[t_0, \dots, t_N]]$ a formal neighborhood of $\Delta(a)$ in $H^*(X;k)$

$\xi \in H^2(X;k)$ $\quad \xi q \partial_q$ is a derivation on $\mathbb{C}[[NE]]$: $\xi q \partial_q\, q^\beta = (\beta \cdot \xi)\, q^\beta$

Then define (H, ∇) over $\mathrm{Spf}\, k[[NE]] \times \mathrm{Spf}\, k[[t_0, \dots, t_N]]$

$\mathcal{H}$ has trivial fiber $H^*(X;k)$ and meromorphic connections

$$\nabla_{\partial u} = \partial_u - \frac{K}{u^2} + \frac{G}{u}$$

$$\nabla_{\partial t_i} = \partial t_i + \frac{T_i *}{u}$$

$$\nabla_{\xi q \partial_q} = \xi q \partial_q + \frac{\xi *}{u}$$

$$K = \Big(c_1(TX) + \sum_{\deg(T_i) \neq 2} \frac{\deg T_i - 2}{2}\, t_i T_i\Big) *$$

$$G = \frac{1}{2}(\deg - \dim X)$$

Fix a nef class $w \in N^1(X)$, then it induces $k[NE] \to k[q]$ by $q^\beta \to q^{\beta \cdot w}$

If w satisfies the following, it further induces $k[[NE]] \longrightarrow k[[q]]$ if

$\forall\, i_1, \dots, i_n, d$, $\exists$ finitely many β s.t. $\beta \cdot w = d$ and $\langle T_{i_1} \cdots T_{i_n} \rangle^{\beta}_{0,n} \neq 0$

$\rightsquigarrow$ logarithmic F-bundle $(H, \nabla)/_{(B,D)}$

Extension of framings theorem

Def: A framing of (H, ∇) is a trivialization of $\mathcal{H}$, s.t. connection matrices have no non-negative u terms, i.e.

$$\nabla = d + \left(\frac{K(t,q)}{u^2} + \frac{G(t,q)}{u}\right) du + \frac{T_i(t,q)}{u} dt_i + \frac{Q_j(t,q)}{u} \frac{dq_j}{q_j}.$$

Given a logarithmic F-bundles $(H, \nabla)/(B, D)$, we can restrict to $t_i = q_i = 0$ and get (H^b, ∇^b) over $\mathrm{Spf}\, k[[u]]$. It has an extra structure of $\nabla_{u q \partial_q}\big|_{t=q=0}$ and $\nabla_{u \partial t_i}\big|_{t=q=0}$ action on $k[[u]]$-mod H^b.

Def. A framing ∇_b^{fr} for $(H, \nabla)\big|_{t=q=0}$ is strong if in this trivialization the $\nabla_{u q \partial_q}\big|_{t=q=0}$ action is independent of u.

Theorem. A framing for $(\mathcal{H}, \nabla)\big|_{t=q=0}$ extends to a framing for $(\mathcal{H}, \nabla)$ iff it's strong. In this case, the extension is unique and explicit.

proof: meaning. extend ∇_b^{fr} to a trivialization of $(H, \nabla)/(B, D)$

$$\nabla = d + \frac{U(t,q,u)}{u^2} du + \frac{T_i(t,q,u)}{u} dt_i + \frac{Q_j(t,q,u)}{u} \frac{dq_j}{q_j}$$

$\rightsquigarrow$ $U(0,0,u)$ has degree u^0, u^1.

$Q_j(0,0,u)$ has degree u^0.

Then $\exists!$ $P \in GL(m, k[[q,t,u]])$ with $P(0,0,u) = \mathrm{Id}$, s.t. $P^* \nabla$ is framed for any t, q, u.

Coro (Uniquess of isomorphism)

If $(H_1, \nabla_1)/(B_1, D_1)$ and $(H_2, \nabla_2)/(B_2, D_2)$ are framed F-bundle, and $(\Phi, f): (H_1, \nabla_1) \longrightarrow (H_2, \nabla_2)$ an isomorphism, then

(1) Φ is uniquely determined by its restriction to b.

(2) if (H_i, ∇_i) $i = 1, 2$, maximal, then f is uniquely determined up to a constant multiple in log direction.

§4. Decomposition of Quantum Cohomology of Projective bundle.

Let X be a smooth projective variety. $E \to X$ vector bundle rk m
$\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}\, \mathcal{E}^\vee)$. Canonical map $\mathcal{O}_E(-1) \to \pi^* E$ over $\mathbb{P}(E)$

$$\begin{array}{c} \mathbb{P}(E) \\ \downarrow \pi \\ X \end{array}$$

Denote by $h = c_1(\mathcal{O}_P(1))$, then we have an iso:

$$\text{iso: } H_{\text{sp}kt} := \bigoplus_{i=0}^{m-1} H^*(X, \mathbb{Q})[-2i] \xrightarrow{\Sigma h^i \cup \pi^*} H^*(\mathbb{P}, \mathbb{Q})$$

Q: $QDM(P) \sim QDM(X)$?

A: Iritani-Koto 23' shows such an isomorphism.

Q: is this unique? How to reconstruct Gromov-Witten invariants of P from X.

Fix w_x ample in $H^2(X, \mathbb{Z})$, basis of $H^*(X.\mathbb{Q})$ extends to $H^*(P, \mathbb{Q})$
$\pi^* w_x$ nef for P, $*$ isn't usual cup product and there is extra enumerative info (computable)

① For P, $QDM(P)$ on $\mathrm{Spf}\, \mathbb{C}[\![q]\!][\![t_{ij}]\!] \times \mathbb{C}_u$

② For $X' = \coprod_{i=1}^{m} X$, $w_x' = (w_x, \cdots, w_x)$, basis of $H^*(X) \rightsquigarrow$ basis of $H^*(X')$

Construct $QDM(X')$ with shifted around $(a_1, \dots, a_n)$, s.t $\Sigma a_i T^i \in H^*(X; \mathbb{k})$.

At $q = t = 0$, it's usual cup product, but $K = c_1 * + \sum_{\deg(T_i) \neq 2} \frac{T_i - 2}{2} a_i T_i$.

Then at $q = t = 0$, compare $QDM(P)|_{/A_u}$ and $QDM(X')|_{/A_u'}$ around a.

Theorem: (HYZZ 24')

For $b' = (b_1, b_2, \cdots, b_m) \in \prod H^*(X)$ s.t. we have an isomorphism

$$\Phi: (H, \nabla) \text{ for } \mathbb{P}\big|_{b=0} \longrightarrow (H', \nabla') \text{ for } X' = \amalg X\big|_{b'}$$

iff the deg $\neq 2$ components of b_i satisfies an explicit equation.

Theorem: the isomorphism $\Phi: (H, \nabla) \longrightarrow (H', \nabla')$ is uniquly determined by $\Phi|_{b=0}$.

Note that ① for $\mathbb{P}$, $c_2 \underset{t=q=0}{*} T = (\pi^* c_1(T_X) + h + \pi^* c_1 V) \cup T + p_* q^* T$

for $p, q : P \times_X P \longrightarrow P$ projection.

② for $\amalg X$, $t = q = 0$, quantum product $* = \cup$

In particular, this isomorphism depends on topological data only.

We can reconstruct the $g=0$, n-pointed GW of $\mathbb{P}(E)$ from X.

§5. Spectral decomposition of F-bundle

2.1. F-bundle

Def (F-bundle): over $B = \mathrm{Spf}\, k[\![t_1 \cdots t_n]\!]$, an F-bundle (H, ∇) is vector bundle over $B \times \mathbb{A}^1_u$, w/ flat meromorphic connections s.t. $\nabla_{u^2\partial_u}$, $\nabla_{u\cdot\partial t_i}$ are regular

Def (maximal): if $\exists\, h \in H_{b,0}$, b is the closed point in B, s.t. the map

$$\begin{aligned} T_b B &\longrightarrow \mathcal{H}_{b,0} \\ \xi &\longmapsto \nabla_{u\xi}|_{u=0}(v) \end{aligned}$$

is an isomorphism. Call $Eu \in TB$ s.t. $\nabla_{uEu}|_{u=0} = K = \nabla_{u^2\partial_u}|_{u=0}$.

Q: $\forall b$, $(H, \nabla)|_{b \times \mathbb{A}^1}$ decompose according to eigenspaces. How can we move in the base so that this decomposition is preserved?

Theorem (Spectral decomposition theorem)

Let (H, ∇) be a maximal F-bundle, and consider $H_{b,0} = \oplus H_i$, s.t. the spectrum of $K_b = \nabla_{u^2\partial u}|_{b,0}$ on H_i are disjoint.
Then $(H, \nabla)/B$ decomposes into a product of maximal F-bundles

$$(H_i, \nabla_i)/B_i \text{ extending } H_{b,0} = \oplus H_i.$$

Idea: (H, ∇) on B can be decomposed into blocks, according to the generalized eigenvalues of $U_{-2}(t=u=0)$. And $\exists$ change of variables

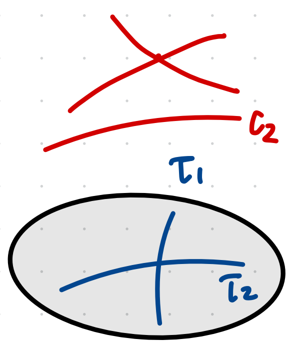
$$t_i \longrightarrow \tau_i(t)$$

s.t. each block $(H^i, \nabla^i)/B_i$ depends only on coordinates of B_i.
Moving in other directions B^j doesn't change eigenvalues of ∇^i.

① If over $k = \mathbb{C}$ and $c_1 \underset{b}{*}$ distinct eigenvalues, then $\tau_i = t - \lambda_i(t, q)$.

② If $k = \mathbb{Q}$, $\det(c_1 \underset{b}{*} - t) = t^2 - 2$, and if $H_{b,0} = H_{\lambda_1} \oplus H_{\lambda_2}$ over $\mathbb{Q}$, then the decomposition extends.

③ If $k = \mathbb{Q}$, $\det(c_2 \underset{b}{*} - t) = t^3$, then it doesn't split.



Proof: ① Base decomposition :

Choose h on B s.t. $\eta = u\nabla|_{u=0}(h): TB \to \mathcal{H}|_{u=0}$ is an isomorphism.
$\rightsquigarrow$ induces a F-manifold structure $(TB, *)$

Further $T_bB = \oplus \eta_b^{-1}(H_i)$ is a splitting of algebra
$\rightsquigarrow (TB, *) = (\oplus D_i, *|_{D_i})$ a decomposition of subalgebra.

By F identity $\Rightarrow [D_i, D_j] \subseteq D_i$ if $j = i$ or 0 if $j \neq i$.
By formal Frobenius theorem, $(B, *) = \prod (B_i, *|_{B_i})$

② Bundle decomposition :

Under $\eta: TB \xrightarrow{\simeq} \mathcal{H}|_{u=0}$, we have $\mathcal{H}|_{u=0}$ decomposition stable $\nabla_{u^2\partial_u}|_u$
and $\mathcal{H}|_{u=t=0} = H_{b,0}$.

We show that this decomposition
extends to whole $\mathcal{H} = \oplus \mathcal{H}_i$, s.t. $\begin{cases} u^2\nabla_{\partial_u}(\mathcal{H}_i) \subseteq \mathcal{H}_i \\ u\nabla_\xi(\mathcal{H}_i) \subseteq \mathcal{H}_i \end{cases}$