Ma 3b Practical – Recitation 4

February 27, 2025

Recall definitions of event space and probability function. Recall Poisson distribution, geometric rv, exponential disttribution, normal and binomial.

Poisson process reference: Section 5.6 of Introduction to Probability-Joseph K. Blitzstein, Jessica Hwang.

Exercise 1. (Possible Variables and Exponential, Gamma distributions.)

We summarize their relation in this exercise by **Theorem 4.2.3. and 4.6.1**[1]

1. Suppose a series of events satisfying the Poisson model are occurring at the rate of λ per month. Let the random variable Y denote the waiting time for the 1st even. Then Y has the exponential distribution

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0$$

2. If Y denotes the waiting time for the rth event, then Y has the Gamma distribution

$$f_Y(y) = \frac{\lambda^r}{(r-1)!} y^{r-1} e^{-\lambda y}, \quad y > 0$$

Now we apply this theorem to the following situation: An Arctic weather station has three electronic wind gauges. Only one is used at any given time. The lifetime of each gauge is exponentially distributed with a mean of one thousand hours. What is the pdf of Y, the random variable measuring the time until the last gauge wears out?

[1] From textbook: Larsen-Marx - An Introduction to Mathematical Statistics and Its Applications

Exercise 2. (Bayes' rule)

Suppose there is a diagnostic test, e.g., CEA (carcinoembryonic antigen) levels, for a particular disease (e.g., colon cancer or rheumatoid arthritis). It is in the nature of human physiology and medical tests that they are imperfect. That is, you may have the disease and the test may not catch it, or you may not have the disease, but the test will suggest that you do.

Suppose further that in fact, one in a thousand people suffer from disease D. Suppose that the test is accurate in the sense that if you have the disease, it catches it (tests positive) 99% of the time. But suppose also that there is a 2% chance that it reports falsely that a someone has the disease when in fact they do not.

What is the probability that a randomly selected individual who is tested and has a positive test result actually has the disease? What is the probability that someone who tests negative for the disease actually is disease free?

The probability of falsely testing positive is much higher compared to that of actually being sick (2

In a real-world scenario, you would most likely only take the test if you suspect that you are actually sick. So the test is not being provided to the entire population, but rather only people who are actually seeking the test. This subset of the population would be biased towards people who are actually sick. So the test only falls apart if everyone gets indiscriminately tested.

Diagnostic test for boneitis.

* 1 in 1000 people are actually sick.
* If you are sick, you will test positive 99% of the time.
* If you aren't sick, you have a 2% chance of testing positive. (False positive)

Q: What's the probability that someone who tested positive is actually sick?

Have: $P(\text{sick}) = \frac{1}{1000} = 0.001 = P(A)$

$P(\text{test pos.} \mid \text{sick}) = 0.99 = P(B|A)$

$P(\text{test pos.} \mid \text{not sick}) = 0.02 = P(B|A^c)$

A = sick
B = test positive

$P(\text{not sick}) = P(A^c) = 0.999.$

WANT: $P(\text{sick} \mid \text{test pos.}) = P(A|B)$

Bayes' law:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{(0.99)(0.001)}{(0.99)(0.001) + (0.02)(0.999)}$$

$$\approx 0.0472 \longrightarrow \boxed{4.72\%}$$