Ma2a Practical – Recitation 5

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Exercise 1. (1st order ODE marathon)
For each of the following IVP, give an implicit equation for the solution. If possible, give a closed-form formula for the solution.

1. $2yy' + 1 = y^2 + x$, with $y(0) = 1$.
2. $y' + xy = y^4$, with $y(0) = 1$.
3. $y' = \frac{x^2 - y^2}{xy}$, with $y(1) = 2$.
4. $y' = \frac{xy - 1}{x^2}$, with $y(1) = 3$.

Exercise 2. (autonomous equation)
Find all the equilibrium solutions of the equation

$$y' = \sin(\pi y),$$

and classify each one in terms of stability. Sketch solution curves in the extended phase space, and describe the behaviour of solutions as $t \to \pm\infty$.

Exercise 3. Consider the differential equation:

$$y' = (x + y + 1)^2 + 3.$$

1. Discuss the uniqueness and existence of solutions given an initial value (x_0, y_0).
2. Solve the equation with the change of variable $z = x + y + 1$.

Solution 1

1. Since $2yy' = \frac{d}{dx}(y^2)$, we make the substitution $z = y^2$ so the equation becomes $z' - z = x - 1$. The homogeneous solution of the 1st order linear ODE is $x \mapsto Ae^x$, for a constant $A \in \mathbb{R}$. Using the method of undetermined coefficients, we find the special solution $x \mapsto -x$. Thus the general solution is $z\colon x \mapsto Ae^x - x$. The initial value $z(0) = 1$ gives $A = 1$. So $z(x) = e^x - x$.

   Thus $y(x) = \pm\sqrt{e^x - x}$, and to determine the sign we use the initial condition $y(0) = 1$. The unique solution of the IVP is $y\colon x \mapsto \sqrt{e^x - x}$.

2. This is a Bernoulli equation, *i.e.* it is of the form $y' + p(x)y = q(x)y^n$. As seen in Recitation 4, the substitution $z = y^{1-n}$ turns this into a linear ODE. Here $n = 4$, so we set $z = y^{-3}$, and $z' = -3y'y^{-4}$. The IVP becomes

$$z' - 3xz = 1, \quad z(0) = 1.$$

   The homogeneous solution of this equation is $x \mapsto Ae^{\frac{3}{2}x^2}$, for $A \in \mathbb{R}$. We find the special solution $z\colon x \mapsto -3e^{\frac{3}{2}x^2}\int_0^x e^{-\frac{3}{2}t^2}dt$. The initial condition $z(0) = 1$ gives $A = 1$.

   Going back to $y = z^{-\frac{1}{3}}$, we obtain that the solution to the IVP is

$$y\colon x \mapsto \frac{1}{\sqrt[3]{e^{\frac{3}{2}x^2}\left(-3\int_0^x e^{-\frac{3}{2}t^2}dt + 1\right)}}.$$

3. We simplify the RHS and rewrite the equation as $y' = \frac{x}{y} - \frac{y}{x}$. Then the equation is seen to be homogeneous, so we set $z = \frac{y}{x}$. Then $y = xz$, so $y' = z + xz'$. Plugging this back into the equation, we obtain

$$\begin{aligned} z + xz' = \frac{1}{z} - z \Leftrightarrow xz' &= \frac{1-2z^2}{z} \\ \Leftrightarrow \frac{zdz}{1-2z^2} &= \frac{dx}{x}. \end{aligned}$$

   The equation is separable, and we integrate it (using the change of variable $u = z^2$) into

$$-\frac{1}{4}\ln|1-2z^2| = \ln|x| + c$$

   We have $z(1) = y(1) = 2$, which gives $c = -\frac{1}{4}\ln 7$. Since $1 - 2z^2 < 0$ for z close to 2, and $x > 0$ for x close to 1, the solution of the IVP in terms of z is defined by

$$\ln(2z^2 - 1) = \ln(7) - 4\ln x = \ln\left(\frac{7}{x^4}\right)$$

Finally, we express in terms of z by taking the exponential

$$z^2 = \frac{1}{2} + \frac{7}{2x^4},$$

and since $z(1) = 2 > 0$ we deduce that the solution is $z \colon x \mapsto \sqrt{\frac{1}{2} + \frac{7}{2x^4}}$. Going back to $y = xz$, we obtain the solution

$$y \colon x \mapsto x\sqrt{\frac{1}{2} + \frac{7}{2x^4}}.$$

4. Method1: the equation is

$$(1 - xy)dx + x^2 dy = 0$$

make it exact.

Method 2: Here, there is no obvious change of variable. We look for a substitution $z = y^n$, as these are the simplest kind of substitution. Then $z' = ny'y^{n-1}$, and the equation can be rewritten as

$$z' = n\frac{zx - z^{\frac{n-1}{n}}}{x^2} = n\frac{z}{x} - n\frac{z^{1-\frac{1}{n}}}{x^2}.$$

Now, we note that for $n = -1$, we obtain a homogeneous equation! So we specialize to $n = -1$, so $z = \frac{1}{y}$, and the equation is

$$z' = \frac{z^2}{x^2} - \frac{z}{x}.$$

We solve the equation by making the substitution $v = \frac{z}{x}$. Then the equation becomes

$$xv' + v = v^2 - v \Leftrightarrow xv' = v^2 - 2v$$
$$\Leftrightarrow \frac{dv}{v^2 - 2v} = \frac{dx}{x}$$

We integrate this (e.g. using the polar decomposition $\frac{1}{v^2-2v} = \frac{1}{2}\left(\frac{1}{v-2} - \frac{1}{v}\right)$) as

$$\ln\left|\frac{v-2}{v}\right| = 2\ln|x| + c$$

Overall, we have $v(x) = \frac{1}{xy(x)}$, so the initial condition translates as $v(1) = \frac{1}{y(1)} = \frac{1}{3}$. We deduce $c = \ln 5$, and inspecting the sign of v around the initial value $x = 1$ allows to lift the absolute values and obtain the equation

$$\ln\left(\frac{2}{v} - 1\right) = \ln 5x^2.$$

Solving for ν we get

$$\nu(x) = \frac{2}{1+5x^2}.$$

Thus

$$y(x) = \frac{1}{x\nu(x)} = \frac{1+5x^2}{2x}.$$

Solution 2

1. Equilibrium positions.
   We solve $y' = 0$ and obtain the constant solutions $y_n = n$, for each integer $n \in \mathbb{Z}$.

2. Stability.
   If n is even, then f is increasing around y_n so the equilibirum is unstable. If n is odd then f is decreasing around y_n, so the equilibirum is stable.

3. Concavity/convexity.
   We study the sign of $y'' = y'f'(y) = \pi\cos(\pi y)\sin(\pi y) = \frac{\pi}{2}\sin(2\pi y)$. It is positive for $k < y < k + \frac{1}{2}$ and negative for $k + \frac{1}{2} < y < k + 1$, where $k \in \mathbb{Z}$.

4. Behaviour of solutions as $t \to +\infty$.
   Let y be a non constant solution to the equation. By the uniqueness theorem, the graph of y cannot intersect the equilibrium positions y_n. So, for each non constant solution there exists an integer $n \in \mathbb{Z}$ such that $y_n = n < y < n+1 = y_{n+1}$. To simplify the discussion, we assume that n is even. Then y is bounded below by an unstable equilibrium, and bounded above by a stable equilibrium. A natural guess is that the solution y converges to the stable equilibirum as $t \to +\infty$. We prove it is indeed the case using the equation and results from analysis.

   Since f is positive on $(n, n+1)$, we have $y' > 0$ so y is strictly increasing. Because y is a continuous function bounded above and strictly increasing, we deduce that it admits a finite limit as $t \to +\infty$, equal to $a = \sup_{t\in\mathbb{R}} y(t)$, and furthermore $n < a \leqslant n+1$. As a consequence, y' admits the limit $f(a) = \sin(\pi a)$ as $t \to +\infty$. We will prove that $f(a) = 0$, which will imply $a = n+1$.

   If $n < a \leqslant n + \frac{1}{2}$, then y' is strictly increasing. In particular, fix a large $T \in \mathbb{R}$ and let $m = y'(T)$. Let $t > T$, we have $y'(t) > m$ so

   $$y(t) = y(T) + \int_T^t y' > y(T) + (t-T)m,$$

   and in particular y is not bounded: this is absurd. Thus we have $n + \frac{1}{2} < a \leqslant n+1$.
   If $n + \frac{1}{2} < a < n$, then y' is strictly decreasing for $t > T$, with T large enough. Thus for $t > T$, we have $y'(t) > m = \inf_{t>T} y'(t)$, and furthermore $m =$

$\lim_{t\to+\infty} y'(t) = f(a) > 0$ since y' is strictly decreasing. By the same reasoning as above, we conclude that in this case the function y is not bounded: this is absurd.

Thus $a = n+1$, and $\lim_{t\to+\infty} y(t) = n+1$. The discussion is similar for n odd, and the solutions converge to the stable equilibirum bounding them below.

5. Behaviour of solutions as $t \to -\infty$.
There are two ways to do this:

   - *Method 1: direct reasoning* based on the monotony of y, similar to the discussion for $t \to +\infty$.
   - *Method 2: change of variable* $s = -t$. Let $u(t) = y(-t)$, then we have $u' = -y'$ so $u' = -\sin(\pi u)$. The interesting thing is that under this substitution, unstable equilibirum and stable equilibirum are exchanged. Then as $t \to +\infty$, the solutions $u(t)$ of the new system tend towards the closest stable equilibirum. Translating back in terms of y, this means that as $t \to -\infty$, the non constant solutions tend to the closest unstable equilibrium.